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Flight Control Laws: Recent Advances in the Evaluation of their Robustness Properties

This paper reviews a set of robustness analysis tools developed by the authors during the last decade to evaluate the robustness properties of high-dimensional closed-loop plants subject to numerous time-invariant uncertainties. These tools are used to compute both upper and lower bounds on the robust stability margin, the worst-case H_∞ performance level, as well as the traditional gain, phase, modulus and time-delay margins. The key idea is to solve the problem on just a coarse frequency grid and to perform a fast validation on the whole frequency range, which results in guaranteed but conservative bounds on the aforementioned quantities. Some heuristics are then applied to determine a set of worst-case parametric configurations leading to over-optimistic bounds. A branch and bound scheme is finally implemented, so as to tighten these bounds with the desired accuracy, while still guaranteeing a reasonable computational complexity. The proposed algorithms are successfully assessed on a challenging real-world application: a flight control law validation problem.

Introduction

Despite recent progress in computer-aided control design techniques, the development of flight control laws remains a challenging task. Even the most sophisticated approaches are still based on simplified models and fail to take all of the requirements into account during the first design phase. As a result, the validation process remains a major and possibly time-consuming issue. There is consequently an obvious need for robustness analysis tools which can be used to perform a reliable and fast preliminary evaluation of the designed controllers before the certification phase.

Among robustness analysis techniques (Lyapunov-based, SOS-based [2], IQC-based [14], …), μ-analysis is now considered as a classical and close to mature approach, which has proved to be useful in many applications (see e.g. [5] and included references). Nevertheless, some technical difficulties remain in specific fields such as robust stability and performance analysis of high-order plants involving largely repeated uncertainties. Indeed, despite recent achievements [19], [20], enhancements are still required to further reduce the conservatism while limiting the required computational time. In this context, the objective of this paper is to describe a set of algorithms and tools thanks to which robustness analysis becomes more reliable and less time-consuming. The ultimate goal is to reduce the number of iterations in a control design process and to avoid expensive Monte Carlo simulation campaigns.

The paper is organized as follows. The problem is first stated in the next section, where all of the key ingredients for μ-analysis are recalled. The robustness margins computation section is then devoted to the characterization of (skew-)μ upper bounds (with a particular emphasis on computational aspects), thanks to which guaranteed robustness margins are obtained. Next, in the worst case analysis section, computational approaches are proposed to determine some enhanced (skew-)μ lower bounds, which are used not only to quantify the conservatism, but also to identify some worst-case configurations useful for design purposes. Extensions of the above results are described in tools extensions section to compute worst-case gain, phase, modulus and delay margins. A branch and bound scheme is also proposed in this section in order to reduce the conservatism. The algorithms are finally illustrated on a challenging aircraft control application in the application to flight control laws validation section. Please note that this is a review paper, which presents the contributions of the systems control group of Onera to the field of μ-analysis. Some sections are thus covered quite briefly, but numerous references are provided throughout the paper.

Problem statement and motivations

Let us consider the standard interconnections of figure 1. $M(s)$ is a stable real-valued linear time-invariant (LTI) plant representing the nominal closed-loop system. Δ is a block-diagonal time-invariant operator, which contains all model uncertainties. For the sake of simplicity (see nevertheless remark 1), only parametric uncertainties are considered in this paper, which means that Δ is a matrix of the form:

$$\Delta = \mathrm{diag}\left(\delta_1 I_{n_1}, \ldots, \delta_N I_{n_N}\right) \qquad (1)$$

where the real scalars δ_i are said to be repeated if $n_i > 1$. Let $n = \sum_{k=1}^{N} n_k$. The set of real $n \times n$ matrices with the same structure as Δ in (1) is denoted by $\boldsymbol{\Delta}$, and $\boldsymbol{B}_\Delta = \left\{\Delta \in \boldsymbol{\Delta} : \bar{\sigma}(\Delta) < 1\right\}$, where $\bar{\sigma}(\bullet)$ denotes the largest singular value.

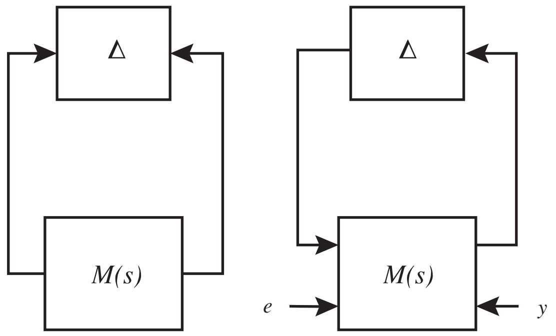

Figure 1 - Standard interconnections for robust stability (left) and robust performance (right) analysis

Two main issues are addressed in this paper: robust stability and worst-case H_∞ performance.

Problem 1 (robust stability) *Compute the maximum value* $k_{\max}$ *such that the interconnection of figure 1 (left) is stable* $\forall \Delta \in k_{\max} \boldsymbol{B}_\Delta$ *, as well as a destabilizing perturbation* $\tilde{\Delta} \in \boldsymbol{\Delta}$ *such that* $\bar{\sigma}(\tilde{\Delta}) = k_{\max}$ *.*

Problem 2 (worst-case H_∞ performance) *Assuming that the interconnection of figure 1 (right) is stable* $\forall \Delta \in \boldsymbol{B}_\Delta$ *, compute the highest value* $\gamma_{\max}$ *of the* H_∞ *norm of the transfer matrix* $F_u(M(s), \Delta)$ *from* e *to* y *when* Δ *takes all possible values in* $\boldsymbol{B}_\Delta$ *, as well as the corresponding value* $\tilde{\Delta}$ *of* Δ*.*

The most efficient technique to answer these two problems is certainly μ-analysis [3]. This is especially true when high-dimensional systems are considered. The underlying theory [17], [5] is not detailed in this paper but a few useful definitions are recalled below.

Definition 1 *Let* ω_i *be a given frequency. If no matrix* $\Delta \in \boldsymbol{\Delta}$ *makes* $I - M(j\omega_i)\Delta$ *singular, then the structured singular value* $(s.s.v.)$ $\mu_\Delta(M(j\omega_i))$ *is defined as* $\mu_\Delta(M(j\omega_i)) = 0$ *. Otherwise:*

$$\mu_\Delta(M(j\omega_i)) = \left[\min\left\{k \in \Re_+ : \exists \Delta \in k\boldsymbol{B}_\Delta, \det\left(I - M(j\omega_i)\Delta\right) = 0\right\}\right]^{-1} \qquad (2)$$

The robustness margin $k_{\max}$ *is then obtained as the inverse of the maximal value* $\mu_\Delta(M(j\omega_i))$ *over the frequency range of interest* Ω *(usually equal to* $\Re_+$ *):*

$$k_{\max} = \left[\max_{\omega_i \in \Omega}\left\{\mu_\Delta(M(j\omega_i))\right\}\right]^{-1} \qquad (3)$$

The exact computation of $\mu_\Delta(M(j\omega_i))$ is known to be NP hard in the general case, but both upper [25] and lower [24], [21] bounds can be determined using polynomial-time algorithms. An upper bound $\bar{\mu}_\Delta(M(j\omega_i))$ provides a guaranteed but conservative value of the robustness margin when Ω is restricted to the single frequency ω_i, while a lower bound $\underline{\mu}_\Delta(M(j\omega_i))$, usually associated with a worst-case parametric configuration $\tilde{\Delta}$, leads to an over-optimistic value.

Computing these bounds over the whole frequency range Ω is a challenging problem with an infinite number of both frequency-domain constraints and optimization variables. It is usually solved on a finite frequency grid $(\omega_i)_{i \in [1,M]}$ and an estimate of the robustness margin is then obtained as:

$$\frac{1}{\max_{i \in [1,M]}\left\{\bar{\mu}_\Delta(M(j\omega_i))\right\}} \le k_{\max} \le \frac{1}{\max_{i \in [1,M]}\left\{\underline{\mu}_\Delta(M(j\omega_i))\right\}} \qquad (4)$$

However, a crucial problem appears in this procedure: the grid must contain the most critical frequency point for which the maximal value of $\mu_\Delta(M(j\omega_i))$ is reached. If not, the upper bound on $k_{\max}$ can be very poor, notably in the case of flexible systems, whose μ plot often exhibits very high and narrow peaks. Even worse, the lower bound can be over-evaluated, $i.e.$ be larger than the real value of $k_{\max}$. Unfortunately, the aforementioned critical frequency is usually unknown. The same difficulty arises when worst-case H_∞ performance is considered. Problem 2 is indeed equivalent to a specific skew-μ problem [4]. Similarly to problem 1, it is thus commonly solved on a frequency grid using polynomial-time algorithms, leading to both lower [16] and (possibly under-estimated) upper [8], [9] bounds on $\gamma_{\max}$.

To overcome the above difficulty, [22] suggests to consider frequency as an additional parametric uncertainty, but this strategy usually leads to a computational burden when applied to high-dimensional systems. In this context, some alternative methods are proposed in this paper to compute both tight and reliable bounds on either $k_{\max}$ or $\gamma_{\max}$:

- Either a μ or a skew-μ upper bound is first computed at some frequency, for which nothing has been assessed yet. This bound is slightly increased, and a frequency interval on which it remains valid is computed. Such a strategy is repeated until the whole frequency range has been investigated, leading to either a lower bound on $k_{\max}$ or an upper bound on $\gamma_{\max}$. The latter is guaranteed over the whole frequency range, and not only on a frequency grid as is the case of most existing methods (see the Robustness margins computation section);
- Some heuristics are then proposed in the Worst case analysis section to determine a worst-case parametric configuration $\tilde{\Delta}$, such that the interconnection between $M(s)$ and $\tilde{\Delta}$ has an eigenvalue on the imaginary axis. Either an upper bound on $k_{\max}$ or a lower bound on $\gamma_{\max}$ is thus obtained. Unlike in most existing methods, frequency is an optimization parameter, which is used to detect critical frequencies and usually leads to more accurate bounds;
- A branch and bound algorithm is finally described in the Accuracy improvements section. It can be used to compute bounds with the desired accuracy and at a reasonable computational cost.

Note also that extensions of the aforementioned methods are proposed in the Unstructured margins section to solve the worst-case unstructured margins problem recalled below.

Problem 3 *(worst-case unstructured margins) With reference to figure 1 (left) and assuming that* $k_{\max} > 1$*, compute the smallest values of the gain, phase, modulus and time-delay margins when* Δ *takes all possible values in* $\boldsymbol{B}_\Delta$ *.*

Robustness margins computation

Computation of a guaranteed stability margin

The classical way [25] to compute an upper bound on $\mu_\Delta(M(j\omega))$ for a given frequency point $\omega=\omega_i$ is to introduce two scaling matrices $D(\omega_i)$ and $G(\omega_i)$, which belong to specific sets **D** and **G** reflecting the block-diagonal structure and the real/complex nature of Δ.

Proposition 1 *Let β_i be a positive scalar. If there are some scaling matrices $D(\omega_i)\in\mathbf{D}$ and $G(\omega_i)\in\mathbf{G}$ such that:*

$$\bar{\sigma}\left(F(\omega_i)^{-\frac{1}{4}}\left\{\frac{D(\omega_i)M(j\omega_i)D(\omega_i)^{-1}}{\beta_i}-jG(\omega_i)\right\}F(\omega_i)^{-\frac{1}{4}}\right)\le 1 \quad (5)$$

where $F(\omega_i)=I+G(\omega_i)^2$ and:

- $\mathbf{D}=\left\{D\in C^{n\times n}, D=D^*>0:\forall\Delta\in\mathbf{\Delta}, D\Delta=\Delta D\right\}$
- $\mathbf{G}=\left\{G\in C^{n\times n}, G=G^*:\forall\Delta\in\mathbf{\Delta}, G\Delta=\Delta^* G\right\}$

then $\mu_\Delta(M(j\omega))\le\beta_i$.

Let us then slightly increase this upper bound, *i.e.* set $\beta_i\leftarrow(1+\varepsilon)\beta_i$, to enforce a strict inequality in condition (5). The key idea is now to compute the largest frequency interval $I(\omega_i)\ni\omega_i$ for which the increased upper bound and the scaling matrices remain valid, *i.e.* such that $\forall\omega\in I(\omega_i)$:

$$\bar{\sigma}\left(F(\omega_i)^{-\frac{1}{4}}\left\{\frac{D(\omega_i)M(j\omega)D(\omega_i)^{-1}}{\beta_i}-jG(\omega_i)\right\}F(\omega_i)^{-\frac{1}{4}}\right)\le 1 \quad (6)$$

As is shown in proposition 2, the determination of $I(\omega_i)$ boils down to a standard eigenvalues computation [20].

Proposition 2 *Let (A_M,B_M,C_M,D_M) be a state-space representation of $M(s)$. Build the Hamiltonian-like matrix:*

$$\boldsymbol{H}=\begin{bmatrix}A_H & 0\\ -C_H^*C_H & -A_H^*\end{bmatrix}+\begin{bmatrix}B_H\\ -C_H^*D_H\end{bmatrix}X\begin{bmatrix}D_H^*C_H & B_H^*\end{bmatrix}$$

where $X=(I-D_H^*D_H)^{-1}$ and:

$$\left[\begin{array}{cc}A_H & B_H\\ C_H & D_H\end{array}\right]= \quad (7)$$

$$\begin{bmatrix}I & 0\\ 0 & \frac{F^{-\frac{1}{4}}}{\sqrt{\beta_i}}\end{bmatrix}\begin{bmatrix}A_M-j\omega_iI & B_MD^{-1}\\ DC_M & DD_MD^{-1}-j\beta_iG\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & \frac{F^{-\frac{1}{4}}}{\sqrt{\beta_i}}\end{bmatrix}$$

Define $\delta_{\omega_{i-}}$ and $\delta_{\omega_{i+}}$ as follows:

$$\begin{aligned}\delta_{\omega_{i-}} &= \max\left\{\lambda\in\Re_-:\det(\lambda I+j\boldsymbol{H})=0\right\}\\ &= -\omega_i \text{ if } j\boldsymbol{H} \text{ has no positive real eigenvalue}\\ \delta_{\omega_{i+}} &= \min\left\{\lambda\in\Re_+:\det(\lambda I+j\boldsymbol{H})=0\right\}\\ &= \infty \text{ if } j\boldsymbol{H} \text{ has no negative real eigenvalue}\end{aligned}$$

Then condition (6) holds true $\forall\omega\in I(\omega_i)$ where:

$$I(\omega_i)=[\omega_i+\delta_{\omega_{i-}},\omega_i+\delta_{\omega_{i+}}] \quad (8)$$

Proof *Let*

$$H(j\omega)=F(\omega_i)^{-\frac{1}{4}}\left\{\frac{D(\omega_i)M(j(\omega_i+\omega))D(\omega_i)^{-1}}{\beta_i}-jG(\omega_i)\right\}F(\omega_i)^{-\frac{1}{4}}$$

The bounds defining $I(\omega_i)$ are obtained by searching for both positive and negative ω of smallest magnitude such that $I-H(j\omega)^*H(j\omega)$ becomes singular, i.e. $\det(I-H(j\omega)^*H(j\omega))=0$*. A state-space representation (A_H,B_H,C_H,D_H) of $H(s)$ is given by equation (7). A state-space representation (A_X,B_X,C_X,D_X) of $I-H(j\omega)^*H(j\omega)$ is then given by:*

$$A_X=\begin{bmatrix}A_H & 0\\ -C_H^*C_H & -A_H^*\end{bmatrix},\quad B_X=\begin{bmatrix}-B_H\\ C_H^*D_H\end{bmatrix}$$

$$C_X=\begin{bmatrix}D_H^*C_H & B_H^*\end{bmatrix},\quad D_X=I-D_H^*D_H$$

Some standard manipulations finally conclude the proof:

$$\begin{aligned}&\det\left(I-H(j\omega)^*H(j\omega)\right)=0\\ \Leftrightarrow\ &\det\left(I+C_X(j\omega I-A_X)^{-1}B_XD_X^{-1}\right)=0\\ \Leftrightarrow\ &\det\left(I+(j\omega I-A_X)^{-1}B_XD_X^{-1}C_X\right)=0\\ \Leftrightarrow\ &\det\left(j\omega I-(A_X-B_XD_X^{-1}C_X)\right)=0\\ \Leftrightarrow\ &\det(\omega I+j\boldsymbol{H})=0\end{aligned}$$

In this context, the following algorithm is proposed to compute a guaranteed robustness margin for a high-dimensional uncertain LTI plant [20], [10], [6]. It mainly consists of a repeated treatment on a list of intervals.

Algorithm 1 *(computation of a lower bound on k_{max})*

1 - Initialization:

- *Define an initial value β_{max} for the μ upper bound, either by a μ lower bound computation (see Section Computation of a destabilizing perturbation) or by setting β_{max} = 0.*
- *Define the frequency range $\Omega=[\omega_{min},\omega_{max}]$ on which the μ* upper bound is to be computed. Let $\boldsymbol{I}=\{I_1\}=\{[\omega_{min},\ \omega_{max}]\}$ be the initial list of frequency intervals to be investigated.

2 - While $\boldsymbol{I}\neq\varnothing$, repeat:

- *Choose an interval $I_i\in\boldsymbol{I}$ and a frequency $\omega_i\in I_i$.*
- *Compute β_i, $D(\omega_i)$, $G(\omega_i)$ such that (5) holds.*
- *Set $\beta_i\leftarrow\max((1+\varepsilon)\beta_i,\beta_{max})$ and apply Proposition 2 to compute $I(\omega_i)$.*
- *Set $\beta_{max}\leftarrow\beta_i$ and update the intervals in $\boldsymbol{I}$ by eliminating the frequencies contained in $I(\omega_i)$.*

3 - A lower bound on k_{max} is given by $k_{LB}=1/\beta_{max}$.

The proposed algorithm is not based on a frequency grid to be defined a priori, with the risk of missing a critical frequency. On the contrary, it relies on a list of frequency intervals which is updated automatically during the iterations. By this approach, the robustness margin is guaranteed over the whole frequency range and no tricky initialization is required.

Remark 1 *Algorithm 1 can be directly applied to the general case where $\Delta(s)$ is a block-diagonal LTI operator not only composed of real scalars (corresponding to parametric uncertainties), but also of complex scalars and unstructured transfer matrices (representing neglected dynamics).*

Computation of a guaranteed H_∞ performance level

The notion of robust performance is of practical importance. Indeed, it is often desirable to quantify the performance degradations, which are induced by model uncertainties and appear before instability. The following proposition is a direct consequence of the main loop theorem [17] and is used to reformulate the worst-case H_∞ performance problem as a specific skew-μ problem. With reference to figure 2, a fictitious complex block Δ_c is added between the output y and the input e. Its size is to be maximized under the constraint that the interconnection is stable $\forall \Delta \in \boldsymbol{B}_\Delta$.

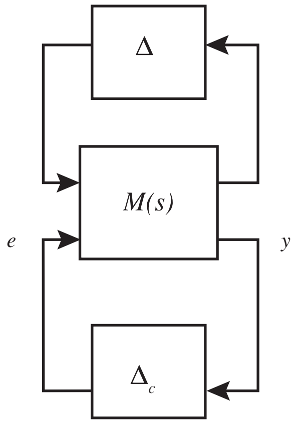

Figure 2 - Augmented system created by addition of a fictitious complex block

Proposition 3 *The following statements are equivalent:*

- $\boldsymbol{F}_u(M(s),\Delta)$ *is stable* $\forall \Delta \in \boldsymbol{B}_\Delta$ *and* $\gamma_{\max} = \max_{\Delta \in \boldsymbol{B}_\Delta} \left\| \boldsymbol{F}_u(M(s),\Delta) \right\|_\infty \leq \gamma$,
- *the size* $\bar{\sigma}(\Delta_c)$ *of the smallest perturbation* $\Delta_c \in C^{p\times p}$ *such that* $\det\left(I - M(j\omega)\operatorname{diag}(\Delta,\Delta_c)\right) = 0$ *for some* $\Delta \in \boldsymbol{B}_\Delta$ *and some* $\omega \in \Re_+$ *is larger than* 1/*γ*,
- $\mu_{\Delta_a}\left(\operatorname{diag}\left(I_{n_r} I_p \sqrt{\gamma}\right) M(j\omega) \operatorname{diag}\left(I_{n_r} I_p \sqrt{\gamma}\right)\right) \leq 1, \forall \omega \in \Re_+$, *where* $\Delta_a = \operatorname{diag}\left(\Delta, C^{p\times p}\right)$.

Proof [17]

Similarly to the previous section, μ_{Δ_a} is replaced with its upper bound $\bar{\mu}_{\Delta_a}$. For a given ω_i, the smallest value of γ such that $\mu_{\Delta_a}\left(\operatorname{diag}\left(I_{n_r} I_p \sqrt{\gamma}\right) M(j\omega_i) \operatorname{diag}\left(I_{n_r} I_p \sqrt{\gamma}\right)\right) < 1$ can then be computed:

- either directly using an LMI characterization [8][9],
- or iteratively using a dichotomy or a fixed-point algorithm together with the formulation (5).

The latter is usually preferred when high-dimensional systems are analyzed, since computational time is much lower. Algorithm 1 (especially step 2b) can thus be slightly modified to compute an upper bound γ_{UB} on the robust H_∞ performance level γ_{max}.

Remark 2 *Algorithm 1 can be further extended to general skew-μ problems, where Δ_c is structured and composed of mixed real/complex uncertainties.*

Worst case analysis

Computation of a destabilizing perturbation

The objective is now to compute an upper bound on $k_{\max}$ to evaluate the conservatism of the lower bound determined in the Computation of a guaranteed stability margin section. This is equivalent to computing a μ lower bound on the whole frequency range. Constructive polynomial-time heuristics exist [24], [21], which provide some worst-case values of Δ. They usually give fast and accurate results when Δ contains some complex uncertainties, but they suffer from two drawbacks. First, convergence problems are often encountered in the purely real case, and the resulting lower bound is then equal to 0. Second, the frequency is fixed. The problem thus has to be solved on a frequency grid, with the risk of missing the most critical parametric configurations even if a fine grid is used. In this context, the key idea of the method described below and initially proposed in [6] is to directly obtain a tight μ lower bound over the whole frequency range rather than at a fixed frequency.

In this perspective, the real μ problem considered in this paper is first regularized by adding a small amount ε of complex uncertainty to each real uncertainty [18]: a perturbation Δ_c is defined with the same structure as Δ, except that the real scalars become complex. The method of [24] or [21] is then applied at a given frequency ω_i, usually with good convergence properties, to the following problem:

$$\begin{aligned} M_a(j\omega_i) &= \begin{bmatrix} M(j\omega_i) & \sqrt{\varepsilon} M(j\omega_i) \\ \sqrt{\varepsilon} M(j\omega_i) & M(j\omega_i) \end{bmatrix} \\ \Delta_a &= \operatorname{diag}(\Delta, \Delta_c) \end{aligned} \qquad (9)$$

The resulting μ lower bound is not a lower bound for the original real μ problem: a perturbation $\Delta_a^0 = \operatorname{diag}\left(\Delta^0, \Delta_c^0\right)$ has been obtained, which renders the matrix $I - M_a(j\omega_i)\Delta_a^0$ singular, but it cannot be claimed that $I - M(j\omega_i)\Delta^0$ is itself singular. Nevertheless, if ε is small enough, an eigenvalue λ_0 of the interconnection of figure 1 (left) is usually located near the point $j\omega_i$ of the imaginary axis.

Starting from this good initial guess $\Delta^0 = \operatorname{diag}\left(\delta_1^0 I_{n_1}, \ldots, \delta_N^0 I_{n_N}\right)$, the last step is to move λ_0 through the imaginary axis to obtain a destabilizing perturbation for the real μ problem. More precisely, in the spirit of [13], a solution is to introduce an additional perturbation $\tilde{\Delta} = \operatorname{diag}\left(\tilde{\delta}_1 I_{n_1}, \ldots, \tilde{\delta}_N I_{n_N}\right)$, which acts as a fictitious feedback gain. The problem is then to find the smallest perturbation $\Delta^0 + \tilde{\Delta}$, which brings the eigenvalue λ_0 on the imaginary axis. As shown in [13], [6], it can be recast as a simple linear programming (LP) problem:

$$\min_{\tilde{\delta}_i} \nu \quad \text{s.t.} \quad \left| \begin{array}{l} -\nu \leq \delta_i^0 + \tilde{\delta}_i \leq \nu \\ \Re\left(\lambda_0 + \sum \alpha_i \tilde{\delta}_i\right) = 0 \end{array} \right. \qquad (10)$$

where:

- $\alpha_i = (uB + tD) \dfrac{\partial \Delta}{\partial \delta_i} (Cv + Dw)$
- (A,B,C,D) is a state-space representation of $M(s)$
- u and v are the left and right eigenvectors associated to the eigenvalue λ_0 of the closed-loop matrix $A_0 = A + B\left(I - \Delta^0 D\right)^{-1} \Delta^0 C$,
- $t = uB\left(I - \Delta^0 D\right)^{-1} \Delta^0$ and $w = \left(I - \Delta^0 D\right)^{-1} \Delta^0 Cv$

In the above formulation, the equality constraint means that the real part of λ_0 must be equal to 0 once the additional perturbation $\tilde{\Delta}$ is applied. An upper bound k_{UB} on $k_{\max}$ is obtained as the minimum value of ν such that (10) holds.

Note that (10) is a linearized version of the problem to be solved. In practice, it may then be necessary to modify it if λ_0 is not sufficiently close to the imaginary axis. In this case indeed, the accuracy of the first order development may not be sufficient, so as to directly move the eigenvalue onto the imaginary axis. A solution consists in partitioning the real segment which separates λ_0 from the imaginary axis, and to iteratively perform the migration on each sub-segment. More precisely, problem (10) is solved ***N*** times, the second constraint being replaced at iteration k with $\Re\left(\lambda_0^k + \sum \alpha_i^k \delta_i^k\right) = \Re\left(\lambda_0\right)\frac{\boldsymbol{N}-k}{\boldsymbol{N}}$, where $\lambda_0^k = \lambda_0^{k-1} + \sum \alpha_i^{k-1}\delta_i^{k-1}$ if $k>1$ and $\lambda_0^1 = \lambda_0$ otherwise.

Note that the aim here is not to compute a μ lower bound at a given frequency, but to directly obtain the highest possible lower bound over the whole frequency range as well as the associated frequency ω^*. Indeed, assume that a bound has been computed for the regularized problem at a frequency ω_i, which is far from ω^*. Using the LP method above, the imaginary axis can however be crossed very close to $j\omega^*$, since no constraint is imposed on the imaginary part of λ_0. Such a behavior is generally observed in practice. It is thus sufficient to apply the algorithm on a coarse frequency grid to obtain a tight upper bound on $k_{\max}$.

Remark 3 *The size of Δ_a in equation (9) is twice the size of the initial matrix Δ. But despite this, computing a μ lower bound usually remains much faster than computing a μ upper bound.*

Worst-case H_∞ performance analysis

In the spirit of the μ lower bound algorithm in the previous section, a two-step procedure is implemented at each point ω_i of a rough frequency grid:

- The unit ball $\boldsymbol{B}_\Delta$ is investigated by iteratively:
  - computing the gradient of $\bar{\sigma}\left(\boldsymbol{F}_u\left(M\left(j\omega_i\right),\Delta\right)\right)$,
  - performing a line search to maximize this quantity (which boils down to computing the eigenvalues of a Hamiltonian-like matrix), until the problem is roughly solved at ω_i.
- Using the value of Δ computed at step 1 as an initialization, a quadratic programming problem, which locally maximizes $\bar{\sigma}\left(\boldsymbol{F}_u\left(M\left(j\omega\right),\Delta\right)\right)$ with respect to both Δ and ω, is repeatedly solved until convergence.

A lower bound γ_{LB} of γ_{max} is finally obtained, as well as the associated value $\tilde{\Delta}$ of Δ [19].

Tools extensions

Unstructured margins

Structured robustness analysis considers that the uncertainties' effects on the system's behavior are perfectly identified, which can be unrealistic. It is thus often desirable to compute worst-case unstructured margins (gain, phase, modulus, time-delay) to evaluate the effective robustness of a system. In this perspective, some additional fictitious uncertainties δ_M corresponding to gain, phase, modulus or time-delay variations are introduced either at the input or at the output of the considered open-loop system $\Sigma(s)$, depending on whether input or output margins are to be computed (see figure 3). Note that either SISO or MIMO margins can be evaluated, δ_M containing either one $\left(\delta_M = \mathrm{diag}(1,\ldots,1,\delta_{M,i},1,\ldots,1)\right.$ or several $\left(\delta_M = \mathrm{diag}(\delta_{M,1},\ldots,\delta_{M,q})\right)$ scalar uncertainties. The expression of δ_M,i is given hereafter for each margin:

- gain margin: $\delta_{M,i} = 1+\hat{\delta}_i, \hat{\delta}_i \in \Re$
- modulus margin: $\delta_{M,i} = 1+\hat{\delta}_i, \hat{\delta}_i \in C$
- phase margin: $\delta_{M,i} = e^{j\varphi_i}, \varphi_i \in \Re$
- time-delay margin: $\delta_{M,i} = e^{-\tau_i s}, \tau_i \in \Re_+$

For gain and modulus margins, the expression of $\delta_{M,i}$ is polynomial and can be written directly in linear fractional form. For phase and time-delay margins, the non-rational elements $e^{j\varphi_i}$ and $e^{-\tau_i s}$ must be transformed first in order to write the interconnection of figure 3 as a linear fractional representation similar to the one of figure 1 (left). For this purpose, the phase variation $e^{j\varphi_i}$ is replaced using the bilinear transformation with $(1-j\hat{\delta}_i)/(1+j\hat{\delta}_i)$, where $\hat{\delta}_i \in \Re$.

This new element, similarly to $e^{j\varphi_i}$, has a unitary modulus and its phase variation covers the whole phase range [-π, π] when $\hat{\delta}_i \in \Re$. For the time-delay margin, the substitution of $e^{-\tau_i s}$ is more delicate because of the Laplace variable s. Nevertheless, it can be replaced by a static rational complex function $f\left(\hat{\delta}_i\right)$, $\hat{\delta}_i \in \Re$, where the variation range of $\hat{\delta}_i$ depends on ω. This dependence can then be treated using some results from [23]. The whole process is detailed in [11].

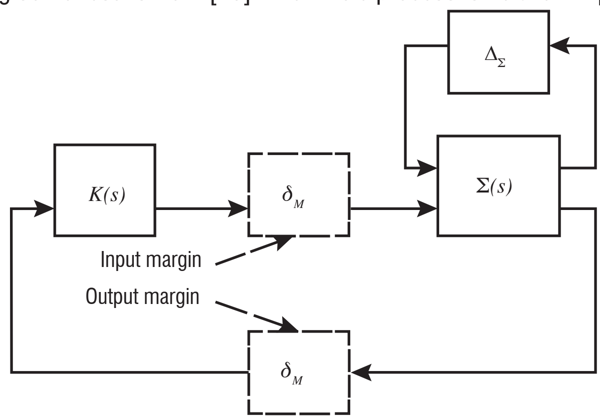

Figure 3 - Introduction of a fictitious uncertainty δ_M

The interconnection of figure 3 can now be written as in figure 1 (left), where Δ contains both Δ_Σ and the $\hat{\delta}_i$. Problem 3 is finally reformulated as follows: assuming that $k_{\max} > 1$, compute the maximum value of the $\hat{\delta}_i$ such that the interconnection between $M(s)$ and Δ is stable $\forall \Delta_\Sigma \in \boldsymbol{B}_{\Delta_\Sigma}$. This is equivalent to a skew-μ problem. Both upper and lower bounds on the various margins can thus be obtained using the algorithms described in the Robustness margins computation and Worst case analysis sections.

Accuracy improvements

Conservatism is defined in this paper as the relative gap η between the lower and the upper bounds on a given quantity x, which can be any of the stability margins or the performance levels considered in the previous sections:

$$\eta = \frac{x_{UB} - x_{LB}}{x_{LB}} \qquad (11)$$

η sometimes reaches unacceptable values, notably in the presence of highly repeated real parametric uncertainties. A well-known technique to ensure that it remains below a specified threshold η_{tol} is to use a branch and bound algorithm [1], [15]. The idea is to partition the real parametric domain into more and more subsets until the relative gap between the highest lower bound and the highest upper bound computed on all of the subsets becomes less than η_{tol}. This algorithm is known to converge for uncertain systems with only real uncertainties [15], *i.e.* conservatism can be reduced to an arbitrarily small value. However, it usually exhibits an exponential growth of computational complexity as a function of the number of real uncertainties. Specifying a threshold η_{tol} is thus used to handle the trade-off between the accuracy of the bounds and the computational time.

Nevertheless, in order to alleviate the computational burden, a strategy based on the progressive validation of the frequency range is proposed here. Assume for example that a subset $\boldsymbol{D}_N$ of the parametric domain and a frequency domain Ω_N are considered at step N of the branch and bound procedure. Algorithm 1 is applied to compute a frequency domain $\Omega_{v,N} \subset \Omega_N$ such that $\det(I - M(j\omega_i)\Delta) \neq 0$ holds $\forall \Delta \in \boldsymbol{D}_N$ and $\forall \omega \in \Omega_{v,N}$. During the next step, the analysis performed on each subset of $\boldsymbol{D}_N$ then only considers the frequencies in Ω_N which have not been validated at step N, *i.e.* which are not contained in $\Omega_{v,N}$. Consequently, after a few steps, the analysis is only restricted to very narrow frequency intervals corresponding to critical frequencies. This results in a drastic reduction of the computational load induced by a classical branch and bound procedure.

Application to flight control laws validation

The algorithms described in the previous sections are now evaluated on a realistic application. All calculations are performed on a 3GHz PC with 3GB RAM.

Description of the model

A high fidelity model composed of 22 states is considered here. It describes both the rigid and the flexible closed-loop longitudinal dynamics of a civilian passenger aircraft. It is parameterized by 4 real parameters characterizing the aircraft's mass configuration: center and outer tanks filling levels δ_{CT} and δ_{OT}, embarked payload δ_{PL} and position of the center of gravity δ_{CG}. The model is written in linear fractional form as shown in figure 1 using the LFR Toolbox for Matlab [12]. As the effects of the parameters on the system behavior are modeled very accurately, the size of Δ is quite large:

$$\Delta = \mathrm{diag}\left(\delta_{CT} I_{48}, \delta_{OT} I_{28}, \delta_{PL} I_{15}, \delta_{CG} I_4\right)$$

Δ is normalized, which means that the whole parametric domain is covered when Δ takes all possible values in $\boldsymbol{B}_\Delta$.

Robust stability analysis

Robust stability is first analyzed in order to check whether stability can be guaranteed over the whole parametric domain. For this purpose, several μ upper and lower bounds are computed, and the results are illustrated in figure 4. The bounds are first computed without branch and bound. A relative gap of about 40% is obtained and the computational time is very reasonable considering the large size of the model. Nevertheless, robust stability cannot be guaranteed over the whole parametric domain, since the upper bound is larger than 1. Much better results are obtained with the branch and bound algorithm, since robust stability can be guaranteed as soon as η_{tol} is less than 15%. Moreover, the additional computational cost induced by the use of branch and bound is quite low thanks to the efficient strategy introduced before used to progressively validate the frequency domain (see Section Accuracy improvements). Note that all μ upper and lower bounds are computed using the algorithms described in Sections Computation of a guaranteed stability margin and Computation of a destabilizing perturbation respectively. Thus, the only tuning parameter in this stability analysis is the threshold η_{tol}.

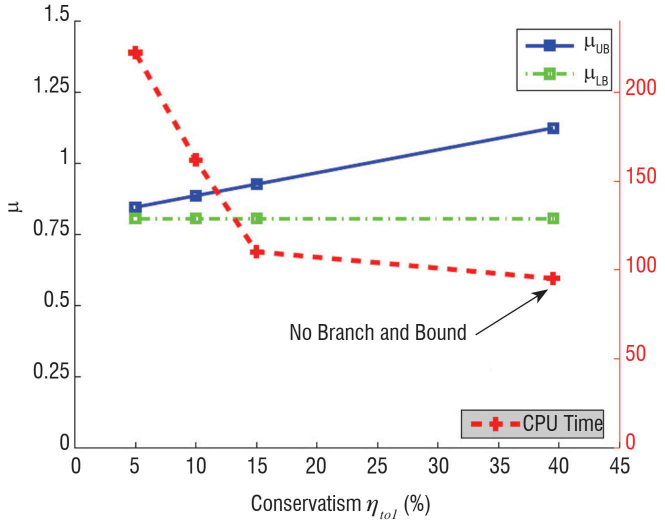

Figure 4 - μ bounds and CPU time versus conservatism

Worst-case H_∞ performance analysis

Worst-case H_∞ performance is evaluated for the transfer function between the vertical wind velocity and the vertical load factor. In order to identify the secondary peaks of the frequency response, the analysis is performed on three contiguous frequency intervals. A skew-μ problem is thus solved on each one of these intervals. Figure 5 shows the bounds on $\gamma_{\max}$ obtained with $\eta_{tol} = 20\%$ as well as the frequency responses of the uncertain system computed on a fine parametric grid. Results are very accurate.

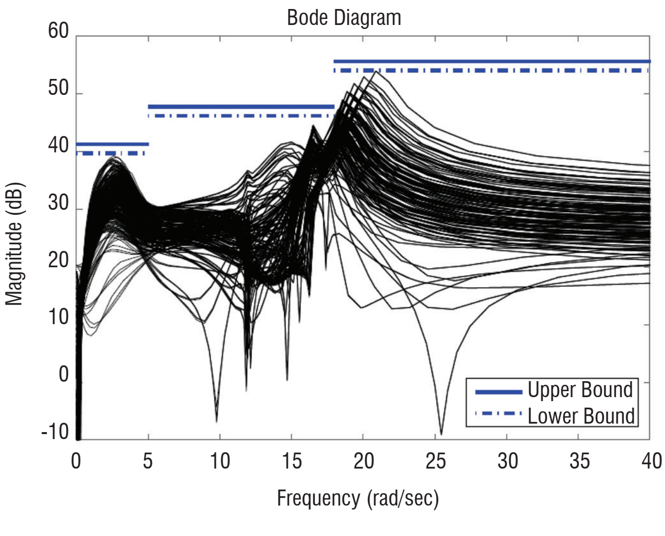

Figure 5 - Bounds on $\gamma_{\max}$ and frequency responses on a fine parametric grid

Worst-case unstructured margins

Worst-case SISO unstructured margins are finally computed. With reference to figure 3, the open-loop system $\Sigma(s)$ is composed of actuators, open-loop aircraft and sensors models in a feedback loop with a dynamic controller K(s). The input of $\Sigma(s)$ is the elevator deflection, while the outputs are the pitch rate and the vertical load factor. Figure 6 shows the bounds on the gain, modulus and phase margins obtained with $\eta_{tol} = 25\%$, and the Nyquist responses of the uncertain system on a fine parametric grid. Once again, results are quite satisfactory and conservatism is efficiently mastered.

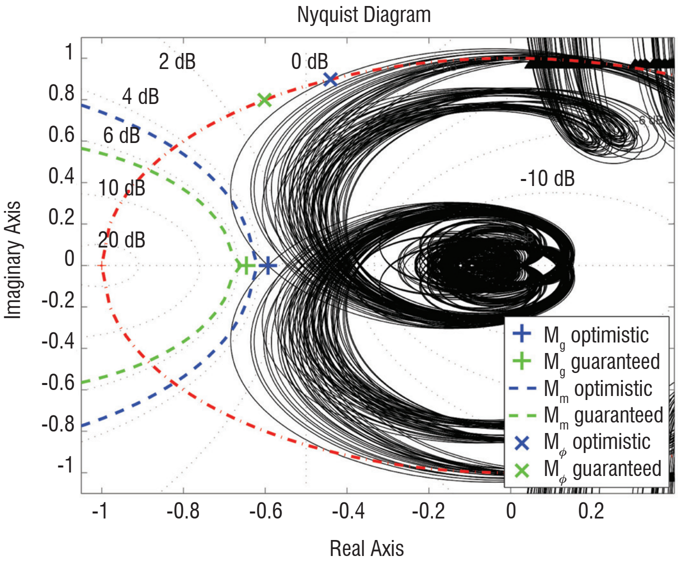

Figure 6 - Upper (optimistic) and lower (guaranteed) bounds on the worst case gain (M_g), modulus (M_m) and phase (M_φ) margins and Nyquist responses on a fine parametric grid

Conclusion and prospects

Several μ-analysis based tools developed by the systems control group of Onera are reviewed in this paper. They are used to compute both upper and lower bounds on the robust stability margin, the worst-case H_∞ performance level, as well as the worst-case gain, phase, modulus and time-delay margins. Unlike most existing methods, these bounds are guaranteed over the whole frequency range, and not only on a finite frequency grid. Moreover, an efficient branch and bound scheme can be used to obtain bounds with the desired accuracy, while still guaranteeing a reasonable computational complexity. These algorithms which will form the basis of a next release of the Skew Mu Toolbox for Matlab [7] should enable to considerably improve the flight control systems validation process in the near future ■

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Acronyms

SOS (Sum Of Square)
IQC (Integral Quadratic Constraint)
LTI (Linear Time Invariant)
MIMO (Multi-Inputs Multi-Outputs)
LFR (Linear Fractional Representation)

Clément Roos holds a PhD in Automatic Control from Supaero, for which he received two awards. He often takes part in industrial projects with Airbus and Dassault, and was involved in the european projects GARTEUR-AG17 and COFCLUO. His research interests focus on LFT modeling, robustness analysis and nonlinear (especially anti-windup) design. He is the author or co-author of several journal and conference papers, as well as a few book chapters.

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