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Stability Analysis by a New Algorithmic Approach Based on Integral Quadratic Constraints: Application to an Aircraft Benchmark

To analyze a non-linear, uncertain and time-varying closed loop representing a fighter aircraft model interconnected with a control law, an Integral Quadratic Constraint (IQC) approach has been used. This approach is particularly interesting for two reasons. The first one is that it is possible with the same stability criterion to analyze a large class of stability problems. The second reason is that the stability criterion is based on frequency dependent inequalities (FDI). Usually, the Kalman-Yakubovich-Popov (KYP) lemma is used, in order to transform this infinite set of inequalities into one linear matrix inequality (LMI). However, this kind of approach leads to a steep increase in the number of optimization variables. Consequently, a new FDI-based algorithmic approach has been developed. Usually, the number of FDI that must be satisfied is infinite or, thanks to a frequency domain gridding, it is possible to avoid this problem but with the drawback of not being able to guarantee the validity of the solution throughout the frequency domain continuum. To tackle this problem, a specific technique has been developed. It consists in computing a frequency domain where the solution is valid. By an iterative approach, this domain is extended to cover $[0,+\infty[$. Thus, the solution obtained from the FDI is necessarily valid throughout the frequency domain continuum and the number of optimization variables remains limited, which makes the IQC approach tractable for high-order models.

Introduction

The IQC technique, which appeared during the nineties, at least in its modern form [13], can be viewed as the merging of two well-known robustness analysis techniques, namely the (scaled) small gain techniques, the best known of these being μ analysis [3], and the positivity/passivity techniques, which study the interconnection of a linear time invariant (LTI) operator with non-linearity (the famous "Lur'e problem"). As a consequence, the IQC technique enables a wide range of problems to be studied, namely the robust stability and performance properties of the interconnection $G(s)-\Delta$ of an LTI operator $G(s)$, with a structured model uncertainty Δ containing non-linearities, LTI and/or linear time-varying (LTV) parameters, neglected dynamics, delays, and specific non-linearities such as friction or hysteresis, etc. The principle is to replace each block of uncertainty by an IQC description of its inputs/outputs, *i.e.*, the inputs/outputs of the block (e.g., a non-linearity inside a sector, possibly with a bound on its slope) are assumed to satisfy a set of Integral Quadratic Constraints [13, 7]. The finer the IQC description of the block is, the less conservative the result will be. This approach is very interesting for two reasons. It includes in the same formalism a large set of linear and non-linear stability theorems. The formalism used in the IQC approach can be described as a unified formalism. Secondly, this unified formalism is based on an input/output approach, namely a frequency domain approach.

In the context of our fighter aircraft application, we use standard IQC descriptions of the uncertainties and focus on the algorithmic issue. Let us recall that the stability criterion of this approach is based on FDI. Thus, the most classical way to solve an IQC analysis problem consists in solving the state-space LMI conditions derived from the KYP lemma, so that the optimization variables come from the IQC multipliers, but also from the Lyapunov matrix P. However, this solution becomes intractable when the order n of the state-space representation becomes too high, since the number of scalar optimization variables in P grows quadratically with n. Note moreover that the initial state-space representation of $G(s)$ is augmented with the state-space representations of the dynamic multipliers, so that even if the order of the initial state-space representation is low, it may increase very fast when introducing dynamic multipliers. Various approaches based on a Hamiltonian matrix have been developed [9, 14, 15] to avoid this problem. Other references propose new multipliers or methodologies to improve results [5, 16, 12, 11]. Here, an alternative technique is implemented.

A first solution is to solve frequency-dependent LMIs, *i.e.*, the FDI, on a frequency domain gridding. However, the main drawback is that it is not possible to guarantee the validity of the solution between the gridding points, even if it is possible to analyze the result *a posteriori* with a very fine frequency domain gridding. However, formally the solution cannot be validated throughout the frequency domain continuum.

A second and more interesting approach, which consists in checking the validity of the solution over the entire frequency domain, has been developed. More precisely, the validation of the solution is done during the optimization problem resolution. In other terms, when the final solution is obtained, this solution is necessarily valid over the entire frequency domain. This validation step is based on a mathematical result for the singular-value maximum of an LFT structure [18] where the Δ block is a real perturbation model. More precisely, when a solution is obtained from a frequency domain gridding, the stability criterion, which depends on frequencies, is written in an LFT form in order to make the frequency ω appear as a real parameter in the Δ block of the LFT [4]. Then, the validity domain of the solution is computed using an algebraic approach. If this domain is $[0,+\infty[$ the solution is valid over the entire frequency domain. Otherwise, frequencies for which the FDI are not satisfied are detected and are added to the initial frequency domain gridding and a new solution is computed with the new gridding, and so on. If no solution is obtained on the gridding, the problem is considered as unfeasible. In brief, the stability problem is recast as an LMI feasibility problem, where the constraints (FDI) are added iteratively. Finally, the number of optimization variables is completely independent from the model order.

However, it remains a problem, since the IQC stability criterion is based on FDI, *i.e.*, a positivity constraint and not a weak gain constraint [18]. However, thanks to a specific bilinear transformation, namely the Cayley transformation [1], this positivity condition, which corresponds to our stability criterion, is replaced by a weak gain condition. By this transformation these two kinds of inequalities are perfectly equivalent. Consequently, it becomes straightforward to evaluate the validity of the solution for the stability criterion.

This is applied to a fighter aircraft interconnected with a control law. The closed loop is written in LFT form, where the Δ block contains one non-linearity, LTV, and LTI parameters. The LTV parameters correspond to the scheduling parameters $Mach$ and V_c (calibrated airspeed), represented by repeated real scalars. Moreover the scheduling parameters are known to be rather slowly time-varying, so that considering arbitrarily time-varying scheduling parameters will lead to conservative results. Thus, the IQC description of time-varying parameters with a bounded rate of variation is used [8]. LTI parameters represent parametric uncertainties of the aircraft model. The non-linearity corresponds to a saturation/dead-zone on the actuator rate output. The objective is to analyze the stability of this non-linear, uncertain and time-varying closed loop.

Notations

Given three operators $P(\cdot)$, $M(\cdot)$ and $\Delta(\cdot)$ of compatible dimensions, the lower and upper Linear Fractional Transformations (LFTs) are respectively defined (for appropriate partitions of P and M) by $\mathcal{F}_l(P,\Delta)=P_{11}+P_{12}\Delta(I-P_{22})^{-1}P_{21}$ and $\mathcal{F}_u(M,\Delta)=M_{22}+M_{21}\Delta(I-M_{11})^{-1}M_{12}$. The star product $\star$ of P and M is defined by:

$$P\star M=\begin{bmatrix}\mathcal{F}_l(P,M_{11}) & P_{12}\left(I-M_{11}P_{22}\right)^{-1}M_{12}\\ M_{21}\left(I-P_{22}M_{11}\right)^{-1}P_{21} & \mathcal{F}_u(M,P_{22})\end{bmatrix} \quad (1)$$

IQC generalities

IQC-based analysis techniques enable us to study a wide range of problems, namely, the robust stability and performance properties of the interconnection $G(s)-\Delta$ of an LTI operator $G(s)$ with a structured model uncertainty Δ containing non-linearities, LTI and/or linear time-varying parameters, neglected dynamics, delays, and specific non-linearities such as friction and hysteresis.

Here, standard IQC descriptions are used for both LTI uncertainties/LTV parameters, Δ, and sector non-linearities, denoted by φ. The originality of our approach resides in the specific algorithm that has been developed to reduce the computational burden. Indeed, standard IQC-oriented analysis methods consist in solving Kalman-Yakubovitch-Popov-based LMI conditions [13]. These standard approaches are, however, intractable for high-order models, since the number of scalar optimization variables quadratically increases with the closed-loop order [2, 19].

An IQC describes a relation between input and output signals of an operator. Since these two formulations are completely equivalent, these constraints can be defined either in the time or the frequency domain. Nevertheless, frequency-domain constraints are often preferred, since this leads to obtaining stability conditions that are easier to handle in comparison with the impulse response for the time domain representation. The definition of an IQC is given in the frequency domain.

Definition 1

Two signals, respectively of dimension m and p, square-integrable over the interval $[0,\infty)$, i.e.: $v\in L_2^m[0,\infty)$, $w\in L_2^p[0,\infty)$, satisfy the IQC defined by $\Pi: jR\to C^{(m+p)\times(m+p)}$, and the Hermitian-valued function, if

$$\int_{-\infty}^{\infty}\begin{bmatrix}\tilde{v}(j\omega)\\ \tilde{w}(j\omega)\end{bmatrix}^{*}\Pi(j\omega)\begin{bmatrix}\tilde{v}(j\omega)\\ \tilde{w}(j\omega)\end{bmatrix}d\omega\geq 0 \quad (2)$$

where $\tilde{v}(j\omega)$ and $\tilde{w}(j\omega)$ respectively correspond to Fourier transforms of v and w, such as $w=\Delta v$.

A priori, the operator Π, called the multiplier, defined from jR in $C^{(m+p)\times(m+p)}$ can be any measurable Hermitian-valued function. In most situations, it is sufficient to use rational operators that are bounded on the imaginary axis.

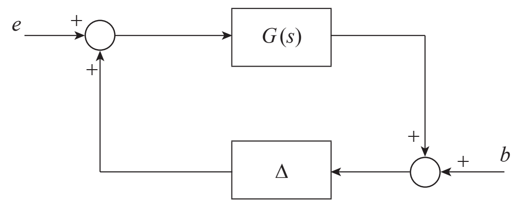

Figure 1 – non-linear, uncertain and time-varying closed loop

The problem consists in analyzing the closed loop that corresponds to the interconnection by a positive feedback of $G(s)$ with Δ, where Δ can be nonlinear and non-stationary. Let us suppose that the input and output signals of Δ satisfy the IQC defined by Π. The following result gives the stability criterion [13].

Theorem 1

Let us assume that $G(s)$ is stable and that Δ is a causal and bounded operator; if

- *the interconnection G-$\tau\Delta$ is well posed for any $\tau \in [0,1]$,*
- *$\tau\Delta$ satisfies the IQCs defined by Π, $\forall \tau \in [0,1]$,*
- *there exists $\varepsilon > 0$ such that:*

$$\forall \omega \in R \quad \underbrace{\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}}_{Z(j\omega)} \leq -\varepsilon I \qquad (3)$$

then, the closed-loop system is stable in the sense of the global asymptotic stability.

It is important to note that if $\tau\Delta$ satisfies several IQC $\Pi_1, \ldots, \Pi_n$, then a sufficient condition for the stability is that $x_1, \ldots, x_n \geq 0$ exist such that the inequality (3) is satisfied for $\Pi = x_1\Pi_1 + \ldots + x_n\Pi_n$, which is a variant of the S-procedure.

The following proposition is very useful to consider the case with several multipliers [6].

Proposition 1

Let us assume a block-diagonal structure $\Delta = diag(\Delta_1, \ldots, \Delta_n)$ and that each Δ_i satisfies the IQC defined by Π_i, where $i = 1, \ldots, n$. Then, Δ satisfies the IQC defined by $\Pi = daug(\Pi_1, \ldots, \Pi_n)$, where the operator $daug$ is defined as follows. If

$$\Pi_i = \begin{bmatrix} \Pi_{i1} & \Pi_{i2} \\ \Pi_{i2}^* & \Pi_{i3} \end{bmatrix} \qquad (4)$$

then

$$daug(\Pi_1, \Pi_2) = \left[\begin{array}{cc|cc} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & \Pi_{21} & 0 & \Pi_{22} \\ \hline \Pi_{12}^* & 0 & \Pi_{13} & 0 \\ 0 & \Pi_{22}^* & 0 & \Pi_{23} \end{array}\right] \qquad (5)$$

List and parameterization of IQCs and stability

In this section, the parameterization of the global multiplier is built to be implemented and solved using the Matlab LMI toolbox.

Sector non-linearities

Let us consider a non-linearity that is memoryless, possibly time-varying, piecewise continuous in t and locally Lipschitz in y. The non-linearity is required that to satisfy a sector condition.

Definition 2

A memoryless non-linearity $\psi : [0,\infty) \times \mathbb{R}^p \to \mathbb{R}^p$ is said to satisfy a sector condition if

$$(w - \underline{k}v)^T (w - \bar{k}v) \leq 0 \qquad (6)$$

where $\underline{k}$ and $\bar{k}$ are gains that represent the limits of the sector and w and v represent the inputs/outputs of the non-linearity.

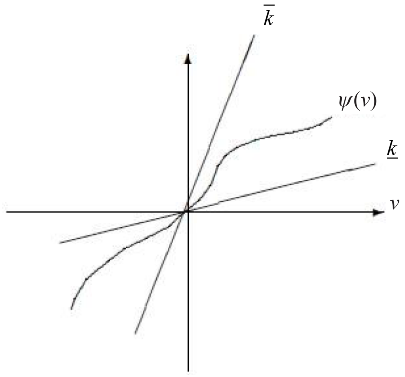

Figure 2 – Sector non-linearity

This definition is illustrated by Figure 2 in the SISO case. The IQC for sector non-linearity φ with a sector (0,1) is the following one [13]:

$$\Pi_\varphi = x\Pi_{sector} + \Pi_{Popov} = \begin{bmatrix} 0 & x + j\omega\lambda \\ x - j\omega\lambda & -2x \end{bmatrix}, \quad x \geq 0, \quad \lambda \in R \qquad (7)$$

Readers can refer to the literature for more details.

Slowly time varying real parameter

Here, $|\delta(t)| \leq 1$, $\dot{\delta}(t) \leq d$, $X = R^*R$ and $Y = S - S^*$. Let us define:

$$R_C(s) = C_R(sI - A_R)^{-1}$$

$$R_B(s) = (sI - A_R)^{-1} B_R$$

$$S_C(s) = C_S(sI - A_R)^{-1}$$

$$S_B(s) = (sI - A_R)^{-1} B_S$$

Let us consider the multiplier that corresponds to LTV parameters [8]:

$$\Pi_{\delta(t)} = \begin{bmatrix} R^*R + d\Gamma^*\Gamma & S - S^* \\ S^* - S & R^*R + d\Upsilon^*\Upsilon \end{bmatrix} \qquad (8)$$

with $\Gamma = \begin{bmatrix} R_B \\ S_B \\ S_C^* \end{bmatrix}$ and $\Upsilon = R_C^* R$. From the parameterization

$$X(j\omega) = \tilde{R}(j\omega)^* U \tilde{R}(j\omega) = R(j\omega)^* R(j\omega)$$

$$Y(j\omega) = V\tilde{S}(j\omega) - \tilde{S}(j\omega)^* V^T = S - S^* \qquad (9)$$

with $U = U^T$, the following multiplier is obtained:

$$\Pi_{\delta(t)} = \begin{bmatrix} \tilde{R}^*U\tilde{R} + d\left(\tilde{R}_B^*\tilde{R}_B + \tilde{S}_B^*\tilde{S}_B + V\tilde{S}_C\tilde{S}_C^*V^T\right) & V\tilde{S} - \tilde{S}^*V^T \\ \tilde{S}^*V^T - V\tilde{S} & -\tilde{R}^*U\tilde{R} + d\tilde{R}^*U\tilde{R}_C\tilde{R}_C^*U\tilde{R} \end{bmatrix}$$

We can clearly see that this multiplier is bilinear in U and V. Then, to satisfy the following inequality

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \left\{ \begin{bmatrix} dV\tilde{S}_C\tilde{S}_C^*V^T & 0 \\ 0 & d\tilde{R}^*U\tilde{R}_C\tilde{R}_C^*U\tilde{R} \end{bmatrix} + \Pi_{lin} \right\} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq 0 \quad (10)$$

with $\Pi_{lin} = \begin{bmatrix} \tilde{R}^* U \tilde{R} + d\left(\tilde{R}_B^* \tilde{R}_B + \tilde{S}_B^* \tilde{S}_B\right) & V\tilde{S} - \tilde{S}^* V^T \\ \tilde{S}^* V^T - V\tilde{S} & -\tilde{R}^* U \tilde{R} \end{bmatrix}$

is not a convex problem.

In order to make this problem convex, let $\Lambda = \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}$ and $W = \begin{bmatrix} \sqrt{d} V\tilde{S} & 0 \\ 0 & \sqrt{d}\tilde{R}^* U \tilde{R}_C \end{bmatrix}$; then, the inequality (10) becomes:

$$\Lambda^* \Pi_{lin} \Lambda + \Lambda^* W W^* \Lambda \le 0 \qquad (11)$$

From this relation, it is possible to use the Schur lemma to obtain the following LMI constraint:

$$\begin{bmatrix} \Lambda^* \Pi_{lin} \Lambda & \Lambda^* W \\ W^* \Lambda & -I \end{bmatrix} < 0 \qquad (12)$$

where Π_{lin} and W are affine in U and V. Of course, if $d = 0$ then $W = 0$, and only the first term of the LMI remains, $\Lambda^* \Pi_{lin} \Lambda$, which corresponds to the constant real scalar.

The global multiplier

In this section, the global multiplier, which corresponds to the general analysis problem, is presented. If φ is the sector non-linearity and $\delta(t) = diag[\delta_1(t) I_n, \ldots, \delta_k(t) I_k]$ the time varying real vector, the closed loop to analyze corresponds to the interconnection of $G(s)$ with $\Phi = diag(\varphi, \delta(t))$. Also, $|\delta_i(t)| \le 1$, $|\dot{\delta}_i(t)| \le d_i$ and $d = diag[d_1 I_n, \ldots, d_k I_k]$. In the case of a constant real scalar it suffices to set $d_i = 0$.

Thus, the global multiplier, which corresponds to Φ, is obtained from Proposition 1:

$$\Pi_{lin} = \begin{bmatrix} 0 & 0 & x + j\omega\lambda & 0 \\ 0 & \tilde{R}^* U \tilde{R} + d\left(\tilde{R}_B^* \tilde{R}_B + \tilde{S}_B^* \tilde{S}_B\right) & 0 & V\tilde{S} - \tilde{S}^* V^T \\ x - j\omega\lambda & 0 & -2x & 0 \\ 0 & \tilde{S}^* V^T - V\tilde{S} & 0 & -\tilde{R}^* U \tilde{R} \end{bmatrix} \qquad (13)$$

The LMI feasibility problem

In brief, the stability of the closed loop, which corresponds to the interconnection by a positive feedback of a sector non-linearity (0,1) and time-varying and/or constant real scalars with a linear part $G(s)$, is ensured by solving the following LMI feasibility problem.

Find $\lambda, x, U = U^T, V$ such that:

$$\begin{bmatrix} \Lambda^* \Pi_{lin}(x,\lambda,U,V) \Lambda & \Lambda^* W(U,V) \\ W(U,V)^* \Lambda & -I \end{bmatrix} < 0 \quad \forall \omega \in R \qquad (14)$$

$$\tilde{R}(j\omega)^* U \tilde{R}(j\omega) > 0 \quad \forall \omega \in R \qquad (15)$$

$$x > 0 \qquad (16)$$

If a solution exists, the closed loop is stable.

<u>Remark 1</u>
This form cannot be directly implemented; a factorized form allowing the dynamic part to be separated into decision variables is involved.

<u>Remark 2</u>
If n_f frequencies are considered, then a problem with $2n_f + 1$ LMI constraints is obtained.

<u>Remark 3</u>
The number of decision variables is completely independent from the closed loop order, but only depends on the structure and the size of the Δ block.

Proposed method

State space approach

The classical approach is based on the Kalman-Yakubovitch-Popov lemma.

<u>Lemma 1</u>
Let us consider M a symmetric matrix, A, B, C, D a state space representation of Φ such as $\Phi(s) = C(sI - A)^{-1} B + D$, and $\forall \omega \in R$ $det(j\omega I - A) \ne 0$; then, the two following propositions are equivalent:
(i) the quadratic constraint

$$\forall \omega \quad \Phi(j\omega)^* M \Phi(j\omega) < 0 \qquad (17)$$

is satisfied

(ii) there exists $P = P^T > 0$ such that

$$\begin{bmatrix} C^T \\ D^T \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} < 0 \qquad (18)$$

The important point is that the second proposition can easily be solved, since it is a feasibility problem under LMI constraints. We notice that the inequality does not depend on the frequency, but a new optimization matrix P has appeared. In other terms, an infinite set of constraints has been transformed into one constraint with a new optimization variable P. To involve the stability criterion (3) in Theorem 1 it suffices to choose a multiplier such as:

$$\sum_{i=1}^{r} x_i \Pi(j\omega) = \Psi(j\omega)^* M \Psi(j\omega) \qquad (19)$$

where M is a symmetric matrix, structured according to the problem considered. This matrix contains a set of optimization variables. With

$$\Psi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} = C(j\omega I - A)^{-1} B + D \qquad (20)$$

the previous lemma allows the inequality (3) to be transformed into an LMI with respect to the optimization matrices P, and M.

Proposed innovative method

An infinite number of LMI constraints has been replaced by one LMI constraint. Nevertheless, this transformation has a major drawback since a new optimization matrix P appears whose size depends on the order of G plus the dynamics of Ψ. More precisely, the number of decision variables grows quadratically, which can lead to a computational problem.

In this paper, the optimization problem is directly solved from frequency-domain inequalities, through a grid-based approach. Of course, the drawback to this approach is the lack of guarantee of the validity of the solution throughout the frequency domain continuum.

To guarantee that the solution is valid over the entire frequency domain, a specific technique based on [4] and [18] is adapted. Also, another advantage is to limit the number of LMI constraints, since only active constraints are added in the LMI optimization problem. Here, the main result is presented:

Let $\Xi = (A_\Xi, B_\Xi, C_\Xi, D_\Xi)$ be the realization of $\Xi(s)$ (of order m), with $\Xi(j\omega) = (I - Z(j\omega))(I + Z(j\omega))^{-1}$ ($(I+Z)$ is invertible) where $Z(j\omega) = Z^*(j\omega)$ is the stability criterion (3), and $\Xi(j(\omega_0 + \delta\omega)) = \mathcal{F}_l(S(\omega_0), \delta\omega I_m)$, with $\forall\, \delta\omega \geq -\omega_0$, *i.e.*, $S(\omega_0)$ is interconnected to $\delta\omega$ as a lower LFT, where $\delta\omega$ is a real parameter. $S(\omega_0)$ is written as

$$S(\omega_0) = \begin{pmatrix} D_\Xi & \dfrac{C_\Xi}{\sqrt{j}} \\ \dfrac{B_\Xi}{\sqrt{j}} & -jA_\Xi \end{pmatrix} \star \left(\frac{1}{\omega_0} \begin{pmatrix} I & I \\ -I & -I \end{pmatrix} \right) \tag{21}$$

<u>Proposition 2</u>

If $\bar{\sigma}(\Xi(\omega_0)) < 1$ then $\bar{\sigma}(\mathcal{F}_l(S(\omega_0), \delta\omega I_m)) < 1$ holds true for $\omega_0 + \delta\omega \in [\underline{\omega}, \bar{\omega}]$, where $\underline{\omega}$ and $\bar{\omega}$ are computed as $\underline{\omega} = \omega_0 + \frac{1}{\eta_n}$ and $\bar{\omega} = \omega_0 + \frac{1}{\eta_p}$, where η_n and η_p are the maximal magnitude real negative and positive eigenvalues of T, respectively, defined as

$$T = \begin{bmatrix} S_{22} & 0 \\ 0 & S_{22}^* \end{bmatrix} - \begin{bmatrix} 0 & S_{21} \\ S_{12}^* & 0 \end{bmatrix} X^{-1} \begin{bmatrix} S_{12} & 0 \\ 0 & S_{21}^* \end{bmatrix} \tag{22}$$

where,

$$S(\omega_0) = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} I & S_{11} \\ S_{11}^* & I \end{bmatrix} \tag{23}$$

<u>Remark 4</u>

When $\bar{\sigma}(\Xi(+\infty)) = 1$, $\bar{\omega} = +\infty \Leftrightarrow \eta_p = 0$, a null eigenvalue is obtained, which means that $\bar{\sigma}(\Xi(\omega))$ crosses the 0 dB axis for $\omega = +\infty$. However, the intersection of the stability criterion with the 0 dB axis has no physical meaning.

<u>Remark 5</u>

The bilinear transformation $\Xi(j\omega) = (I - Z(j\omega))(I + Z(j\omega))^{-1}$ with $(I+Z)$ invertible allows a positivity condition to be transformed into a weak gain condition:

$$\bar{\sigma}(\Xi) \leq 1 \Leftrightarrow Z + Z^* \geq 0 \tag{24}$$

In brief, if we consider a transfer matrix Ξ, in order to determine the frequency domain containing ω_0 such as the maximal singular value of $\Xi(j\omega)$ is inferior to 1, it suffices to evaluate $\bar{\omega}$ and $\underline{\omega}$ as above.

In the iterative approach, proposed in Algorithm 4.2.1, the validation step is performed *a priori* and during the LMI optimization problem resolution. The choice of the initial grid has no influence on the feasibility problem. It is possible to choose a singleton at the first iteration. However, in order to limit the number of iterations, and consequently the calculation time, without any *a priori* knowledge, it is recommended to take some frequencies roughly spread over the frequency domain. It is possible, when first solutions are obtained, to tune this initial frequency grid, in order to decrease the number of iterations.

Sketch of the algorithm

The algorithm can be summarized by the following steps:

Algorithm 1: Iterative IQC resolution

Data: $G(j\omega)$ the stable fixed block of the LFR, multiplier $\Pi(\omega_i)$ and $\omega_i \in R_+$, $i = 1, \ldots n_f$.
Result: A stability proof of the LFR model, including nonlinear sector saturations.
while *Stability not checked* **do**

For $i = 1, \ldots n_f$, check the stability criterion

$$\begin{bmatrix} G(j\omega_i) \\ I \end{bmatrix}^* \Pi(\omega_i) \begin{bmatrix} G(j\omega_i) \\ I \end{bmatrix} < 0. \tag{25}$$

if (25) *has solutions* **then**

- Set $\Pi_i \leftarrow \Pi(\omega_i)$ be the solution obtained at ω_i.
- Set $\omega_0 \leftarrow w_i$ and apply Proposition 2.
- For each solution Π_i, a frequency-domain $\Omega_i = [\underline{\omega}_i, \bar{\omega}_i]$ is obtained. $\Omega_{valid} = \bigcup_{i=1,\ldots,n_f} \Omega_i$.

if $\Omega_{valid} = [0 + \infty)$ **then**

The solution composed by the set of Π_i is validated over the entire frequency domain.
<u>Stability is proved</u>, **stop**.

else

- Determine the complementary set $\Omega_{novalid} = C_{[0,+\infty)} \Omega_{valid}$.
- Select one or several frequencies in $\Omega_{novalid}$ and update the grid.

<u>Stability cannot be proved</u>, **stop**.

This algorithm is a specific case of outer approximation algorithms [17, 14]. Of course, if no solution is obtained on the finite set of frequencies, the problem is considered as infeasible. If a solution is obtained, necessarily this solution is valid over the entire frequency domain.

<u>Remark 6</u>

The stability condition is checked as described in Section 4.2, where critical frequencies are added iteratively. However, it is necessary to check the positivity constraint of the multiplier $X(j\omega) = \tilde{R}(j\omega)^* U \tilde{R}(j\omega)$ for all ω. Of course, it is possible to proceed in the same way: frequencies for which the multiplier is negative are added in the optimization problem by an iterative approach. However, this approach is useless and increases the computational burden. By noting that X is hardly ever positive when the stability condition is satisfied for any frequency, the positivity condition of X is checked by the technique presented previously just once at the end of the algorithm, i.e., when the stability criterion is satisfied over the entire frequency domain. In the exceptional case where a frequency exists such that $X(j\omega) < 0$, then this frequency is added in the optimization problem and another solution is sought, to satisfy the stability condition and the positivity of X.

Remark 7

It is possible by this approach to solve directly over the frequency domain without any approximation based on rational functions. A solution based on irrational multipliers is proposed in [19].

Application

The objective is to analyze the stability problem with a fighter aircraft model. For the problem considered, the critical static non-linearity is the rate limiter, which has been transformed into a dead zone. In brief, the analysis is performed as follows. For specific values of the speed variation d, a sector size $(0,k)$ is determined by a dichotomic approach, so the closed loop stability is guaranteed. This analysis is performed for two cases essentially: one case with one static non-linearity, which corresponds to the rate limiter, and another case with one non-linearity and all other LTV and LTI parameters. The LTV parameters are the Mach number and calibrated air speed. These two time-varying parameters represent the flight case. The Mach number and the calibrated airspeed, respectively, vary from 0 to 1 and from 150 to 275 kts. The nominal rate of variation is $d = 0.2$ for both. The LTI parameters correspond to real uncertainties on the model. This LFT real uncertainties are a combination of various physical real uncertainties, such as mass, center of gravity position, etc. This transformation is necessary to obtain a limited size for the LFT model. In brief, it is not possible to associate a real uncertainty of the LFT model to a physical parameter of the aircraft model. However, the important fact to keep in mind is that the stability analysis is performed for the maximum variation of real uncertainties and not for a restricted domain. In other words, if the stability is guaranteed with all LTI parameters for all possible uncertainty values, the stability is guaranteed for the entire domain of physical parameters. Of course, this kind of transformation can lead to a difficult interpretation if the analysis is performed over a restricted domain for LTI parameters, since it is not possible to easily link this restricted domain to the physical domain; however, this is not the case here.

Another and last point is the following one. The sector size is determined by the value k. However, this value can be interpreted as an amplitude of the non-linearity input u. Since the problem has been normalized, for a rate limiter of 80 deg / s, $u = 1$ corresponds to 80 deg / s and, more generally, to a physical signal equal to u*80 deg / s, For example if $k = 0.5$, this value corresponds to a normalized non-linearity input $u = \frac{1}{1-k} = 2$. It means that, for any normalized $u \le 2$, or equivalently 160 deg / s, the stability of the non-linear closed loop is ensured. In other words, if any realistic pilot order, turbulence or discrete gust leads to having $u \le 2$ for the normalized signal, the stability is ensured. If there exists a realistic input signal that leads to having $u > 2$ for the normalized non-linearity input, then the stability cannot be guaranteed. In brief, the stability can be interpreted as an input/output approach: for any bounded input signal; any output signal of the non-linear closed loop is bounded. This bound corresponds to classical norms of signals, as l_1, l_2, l_∞,.... To be complete, the case where $k = 1$ corresponds to an infinite stability domain.

The parameter occurrence is the following: 1 for the rate limiter, 2 and 8 for the 2 LTV parameters, and 1,1,1,1 and 1 for the 5 LTI parameters. The dynamic for the multipliers $\tilde{R}$ and $\tilde{S}$ is a first order low-pass filter with a pole at 10 rad/s.

Analysis with one rate limiter

In this section, the analysis is performed with just one non-linearity of sector $(0,1)$. This case is interesting for several reasons:

- This case represents the best result that can be expected, since the size of the sector decreases with the number of LPV and LTI parameters that are considered in the analysis problem.
- This case can be interpreted very easily using a SISO representation like the Popov and Nyquist plot.

Let us recall that the Popov plot represents the plot $Re[G(j\omega)]$ versus $\omega Im[G(j\omega)]$. The closed loop stability is ensured if the Popov plot of $G(j\omega)$ lies to the right of the line that intercepts the point $-1+0j$ with a slope $1/\lambda$ [10].

- The size of the sector is consistent with the Popov and the Nyquist plot, since the value obtained for the sector leads the Nyquist and the Popov plot to be very close to the critical point -1.

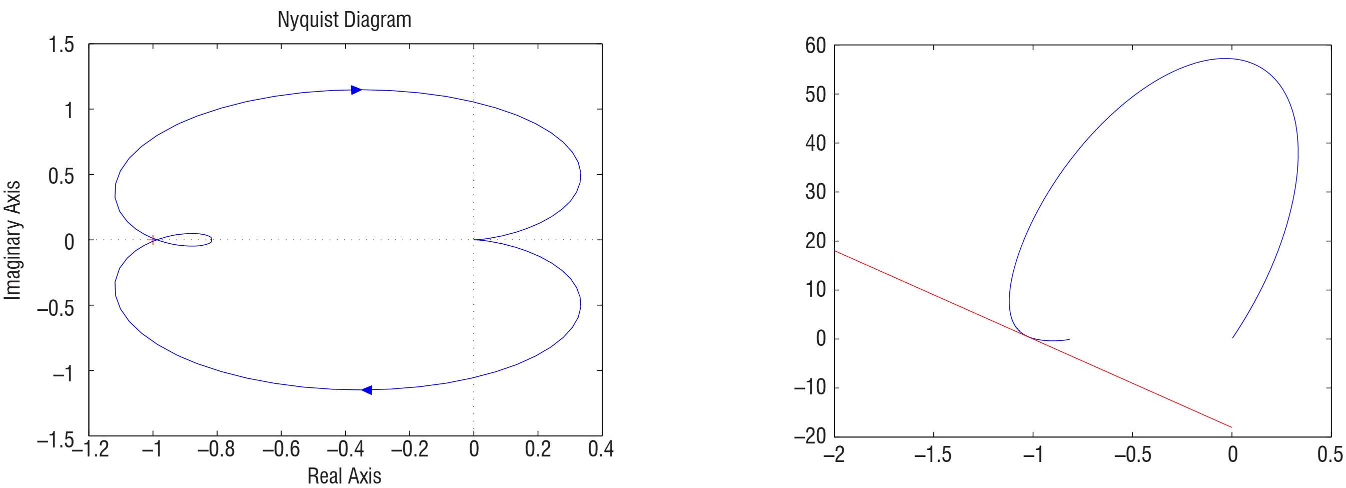

Figure 3 – Popov plot

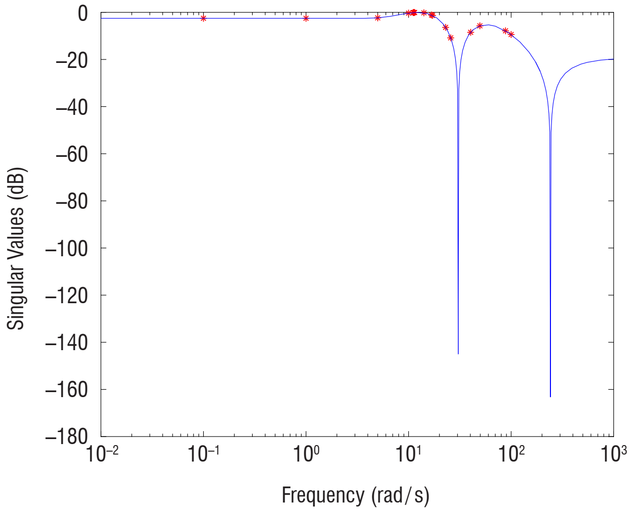

Figure 4 – Singular values

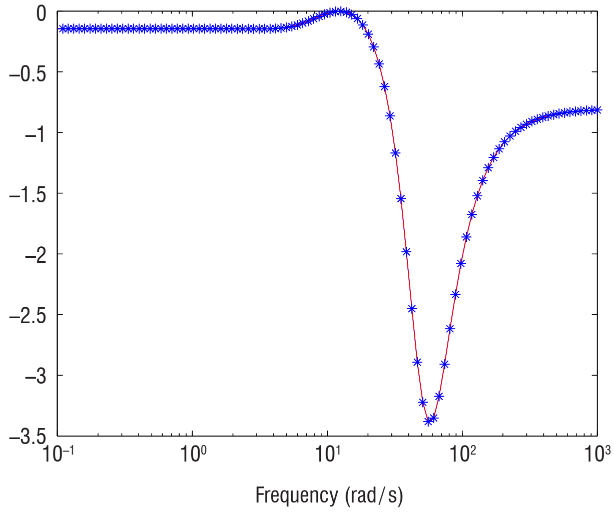

Figure 5 – Eigenvalues

- The solution obtained by the IQC approach is consistent with the graphical interpretation, since we note that the Popov plot lies to the right of the red line whose slope is 1 / λ.

- To confirm the solution, the eigenvalues of the stability criterion and singular values of the transformed criterion by the Cayley method are respectively negative and inferior to $k = 0.81 \Leftrightarrow u = 5.26$.

The limit of the sector is $k = 0.81 \Leftrightarrow u = 5.26$. Of course, this size is determined for the nominal case, *i.e.*, without uncertainties and LTV parameters.

Analysis with one rate limiter, LTV Mach number, and five real uncertainties

This case is interesting, since it combines three kinds of problem: a static non-linearity, LTV parameter and LTI real uncertainties. It is the typical problem treated by the IQC approach. In addition, since the Δ block is not very large, the computational burden is very limited and it is possible to perform several simulations with different bounded rates d to test the algorithm. The LTV parameter considered is the Mach number. Finally, the Δ block has 8 inputs/outputs.

	LTI	d=0	d=0.1	d=0.2	d=0.5	d=0.8	d=10	d=100	d=1000	X,Yconstant
k	0.89	0.89	0.86	0.85	0.85	0.84	0.81	0.80	0.74	0.78
u	9.09	9.09	7.14	6.67	6.67	6.25	5.26	5.00	3.85	4.55

From this table, we can make several remarks:

- For the LTV case, we note that the size of the sector decreases with the rate of variation, which is consistent physically and mathematically since d appears as a penalty in the LMI constraints.

- The LTV case with $d = 0$ is equivalent to the LTI case, from a mathematical point of view. However, it is important to check this point from an algorithmic point of view. The results are the same, or very similar.

- The LTV case with $d \to \infty$ is equivalent to the case where the multipliers X and Y are chosen as constant. Indeed, as indicated previously, if the rate of variation is arbitrary high, the multiplier must be constant. In brief, the size of the sector is evaluated in the LPV context with d = 100 or 1000 and compared to the solution obtained with constant multipliers. The results are very closed in terms of sector limits.

To complete these previous results, the stability analysis is performed when the LTV calibrated air speed V_c is replaced by a specific value and d = 0.2. Of course, the nominal case is represented by the value 0.

	V_c=+1	V_c=+0.5	V_c=0	V_c=–0.5	V_c=–1
k	0.81	0.83	0.85	0.80	0.77
u	5.26	5.88	6.67	5.00	4.35

Analysis with one rate limiter, LTV Mach number and airspeed, and five real uncertainties

This system represents the final case, which combines all LTV and LTI parameters. The Δ block has 16 inputs/outputs.

Two results have been obtained: for the LTI case (d = 0) and the nominal case (d = 0.2).

- With d = 0 the sector obtained is $k = 0.45 \Leftrightarrow u = 1.82$.

- With d = 0.2 the sector obtained is $k = 0.31 \Leftrightarrow u = 1.45$.

In this last case, the number of optimization variables is 309, with some dozens of constraints. The calculation time varies from a few to ten minutes.

Conclusion

In this article, a new algorithmic approach based on IQC technique has been presented. Usually, the KYP lemma is used to transform the stability criterion, which consists in an infinite set of LMI constraints, into one LMI. The main drawback is that a decision matrix P is added and consequently the number of decision variables grows quadratically. To avoid this kind of problem, the frequency domain criterion is explicitly used. To guarantee the solution over the entire frequency domain and not only over the frequency domain gridding, an LFT structure is involved where Δ contains the frequency ω. Then, it becomes possible to treat this variable as a continuous variable, as for any μ analysis problem, and to check the validity of the solution over the frequency domain continuum. In brief, the number of decision variables is independent from the order of $G(s)$ and the solution obtained using the gridding is valid over the entire frequency domain. Finally, to illustrate the approach, this is applied to a fighter aircraft. This application presents one sector non-linearity, two LTV parameters, which correspond to the flight case, and five real uncertainties. The results show that the algorithm is effective for dealing with a large class of stability analysis problems ■

Acronyms

KYP	(Kalman-Yakubovich-Popov)
FDI	(Frequency Dependent Inequalities)
LTI	(Linear Time Invariant)
LTV	(Linear Time-Varying)
IQC	(Integral Quadratic Constraints)
LMI	(Linear Matrix Inequalities)
LFT	(Linear Fractional Transformation)

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