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Robust consensus-seeking via a multi-player nonzero-sum differential game

Considering a class of linear multi-agent systems, we study the problem of consensus-seeking in the presence of an exogenous signal, possibly representing a disturbance. We formulate the *robust consensus-seeking* problem as a nonzero-sum differential game. Exact solutions are characterized by a system of coupled differential Riccati equations (in the finite-horizon case) or algebraic Riccati equations (in the infinite-horizon case), whereas approximate solutions are characterized in terms of coupled matrix *inequalites*. Simulations for two examples are presented to illustrate the resulting performances: one example concerns consensus among agents described by single-integrator dynamics and the other concerns formation flight of unmanned aerial vehicles.

Introduction

Multi-agent systems have gained interest for a wide range of applications including, but not limited to, robotics (see, e.g. [11] and references therein), opinion dynamics (see, e.g. [6] and references therein) and power systems (see, for instance, [12], [22]). Notably, in part due to recent technological advances related to unmanned aerial vehicles (UAVs) and small satellites, multi-agent systems play a major role in a large variety of aerospace applications. For instance, the control of relative distances and orientations between multiple spacecraft to achieve a desired formation is considered in [17]. The spacecraft formation-following problem is also considered in [20], where a graph theoretic formulation of the leader-following approach (introduced in [39]) is provided and solved by means of linear matrix inequalities.

Consensus-seeking for multi-agent systems describes problems in which agents are required to "reach an agreement" on a certain value and is a particularly active research domain, see e.g. [32]. Consensus control aims at driving the states of all agents to reach a common value and plays a major role in various applications, such as formation flight, cooperation in networks, and fault detection and identification (see, for instance, [27], [32], and [34]). In aerospace applications, this topic is of particular interest given that several important problems, such as synchronization and formation control, can be formulated as (dynamic) consensus problems, as seen for instance in [26], [33], [41], [42], [43]. The performances of any consensus protocol are basically sensitive to the presence of persistent perturbations or potential information failures. Various works, such as [31] and [40], have been dedicated to increasing resilience of the controlled systems against various perturbation sources.

Many of the aforementioned consensus-based approaches address the issue of determining control actions for individual agents in a *distributed* manner (e.g., based on neighbor-to-neighbor communication). However, important aspects such as robustness and/or optimality are often recognized, but not addressed (see, e.g. [38]). In this paper we consider the problem of *robust consensus-seeking*, namely the problem of seeking consensus among members of a multi-agent system in the presence of disturbances. In particular, we consider a class of multi-agent systems, described by linear dynamics, and study the problem of consensus control in the presence of an exogenous input, representing a disturbance. The main contribution of this paper is the formulation of the robust consensus-seeking problem as a nonzero-sum differential game with *multiple* players – a formulation which, differently from most existing results concerning consensus problems, enables the consideration of scenarios in which agents are influenced by uncertain and unmodeled perturbations. As will be demonstrated in this paper, game theory provides a convenient framework to evaluate the discrepancy resulting from antagonist environments, and to define a reactive control that provides a suitable compromise between performance and robustness. The motivations of the game-theoretic formulation are twofold. Firstly, since the game theoretic framework essentially models strategic decision making, it allows for the elegant characterization of possibly conflicting goals (such as, for instance, robustness and optimality). Secondly, recent developments in the control engineering field indicate that game theory can serve as a useful tool to systematically design distributed controllers. Although promising preliminary results are available (see, e.g. [8], [9], [16], [21]), the issue of distributed control design is not addressed in this paper, since game theory-based approaches to distributed control design have yet to be fully developed. Once the results are more mature, the formulation of the consensus-seeking problem as a nonzero-sum differential game provided in this paper can be more readily integrated with game theory-based methods for distributed control design in the future. Moreover, by capturing the performance – in terms of optimality *and* robustness – of the closed-loop system in the absence of communication constraints, the results presented herein may constitute a benchmark for new distributed control methods.

Concerning robustness, it is well-known that H_∞ control can be considered as a two-player zero-sum differential game (see e.g. [4]). Concerning robustness *and* optimality on the other hand, mixed H_2/H_∞ control cannot be described using the same framework due to the inherent trade-off between the two objectives. To reflect the presence of this trade-off, in [18] the classical mixed H_2/H_∞ control problem has been formulated as a two-player *nonzero-sum* differential game. Whereas linear systems are considered in [18], the nonlinear counterparts of the differential game formulation of mixed H_2/H_∞ control have been explored in [19], [25].

Differently from the framework considered in [18], where the control problem involves a *single* optimization criterion and a single robustness criterion, herein we consider a setting with *multiple* optimization criteria. This formulation is adopted to reflect practical scenarios in which each agent has an *individual* objective (e.g., to reach a particular position relative to neighbors in an optimal manner), while it is desired for the system, as a whole, to satisfy a certain robustness property. More precisely, the overall multi-agent system is represented by a dynamical system with several inputs – one for each agent – and each agent is associated with an individual cost functional designed to encourage consensus with neighboring agents in the presence of a disturbance.

The remainder of the paper is organized as follows. The robust consensus-seeking problem is defined and formulated as a nonzero-sum differential game in § "Robust consensus-seeking". Exact solutions of the differential game are characterized, both in the finite-horizon and infinite-horizon cases, in terms of coupled differential Riccati equations and coupled algebraic Riccati equations (AREs), respectively, in § "Exact solutions ". Noting that solutions to the coupled AREs, which arise in the context of infinite-horizon differential games may, in general, be difficult to obtain, approximate solutions to the differential game are characterized by means of matrix inequalities (instead of the AREs) in § "Approximate solutions". Simulations corresponding to two examples are presented in § "Simulations" to illustrate the performances of the resulting controllers. One example, presented in § "Consensus at the origin", concerns consensus-seeking among agents described by single-integrator dynamics. The second example concerns a problem of formation flight of UAVs and is presented in § "Application to UAV formation flight". Finally, some concluding remarks are provided.

Notation

Standard notation is adopted throughout this paper. $\mathbb{R}$ denotes the set of real numbers, whereas $\mathbb{C}$ denotes the set of complex numbers and $\mathbb{C}^-$ denotes the open left-half complex plane. Given a square matrix $M \in \mathbb{R}^{n\times n}$, its spectrum is denoted by $\sigma(M)$. The identity matrix is denoted by I. Given a vector $v \in \mathbb{R}^n$, its Euclidean norm is denoted by $\| v \|$.

Robust consensus-seeking: A differential game formulation

Consider a set of $N > 1$ agents, where the dynamics of each agent i, $i = 1,\ldots,N$, is described by

$$\dot{x}_i = A_i x_i + B_i u_i + B_{N+1}\omega \tag{1}$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector of Agent i, $u_i(t) \in \mathbb{R}^m$ is its control input, $\omega \in \mathbb{R}^r$ is an exogenous input representing a disturbance or perturbation common to all agents, and $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, for $i = 1,\ldots,N$, and $B_{i+1} \in \mathbb{R}^{n\times r}$ are constant matrices.

The individual states x_i, $i = 1,\ldots,N$, can be combined (in a manner to be specified) to form a *global state* $X \in \mathbb{R}^{Nn}$ with the *global system* described by linear dynamics of the form

$$\dot{X} = A^g X + B_1^g u_1 + \ldots + B_N^g u_N + B_{N+1}^g \omega \tag{2}$$

where the matrices A^g and B_i^g, $i = 1,\ldots,N$, are specified according to the definition of the global state. The global state could, for instance, be defined as the simple aggregate of all individual states (as considered in § "Consensus at the origin") or in terms of an error variable, for example, in terms of relative differences between neighboring agents.

In this paper we consider the case in which each agent seeks to reach a consensus with its neighboring agents subjected to the disturbance ω. The connectivity between agents is described by a directed graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, where $\mathcal{V} = \{1,\ldots,N\}$ is the set of vertices and $\mathcal{E}$ is the edge set. Each vertex corresponds to an agent and, if $(j,i) \in \mathcal{E}$, Agent j is said to be a neighbor of Agent i. Since we consider directed graphs, $(j,i) \in \mathcal{E}$ does not necessarily imply $(i,j) \in \mathcal{E}$. We assume that a connection between two agents, *i.e.,* $(j,i) \in \mathcal{E}$, implies that the i-th Agent has (through measurements or some form of communication) access to the state x_j of the j-th Agent. Let $\mathcal{N}_i$ be the set of neighbors of Agent i and let N_i denote the cardinality of the set, *i.e.,* $N_i = |\mathcal{N}_i|$. In the following we adopt the convention that Agent i is included in its own neighbor set only if explicitly stated, *i.e,* if $(i,i) \in \mathcal{E}$, for $i = 1,\ldots,N$. Let $e_i(t)$ denote the consensus error of Agent i, namely e_i is given by

$$e_i = \left(x_i - \frac{1}{N_i} \sum_{j\in\mathcal{N}_i} x_j \right) \tag{3}$$

and let $\| Z(t) \|^2$ denote a penalty variable given by

$$\|Z\|^2 = \sum_{i=1}^{N} \|e_i\|^2 .$$

In the remainder of this paper, u_i denotes a feedback control strategy, namely $u_i = u_i(x(t))$, for $i = 1,\ldots,N$. The robust consensus-seeking problem is defined as follows.

Problem 1: robust consensus-seeking
Consider a system described by the dynamics (2). Determine feedback control laws u_i^*, $i = 1,\ldots,N$, such that the following conditions hold.

(C1) When the worst-case disturbance $\omega^*(x(t))$ and the control actions u_j^*, $j = 1,\ldots,N$, $j \neq i$, are applied, u_i^* is such that the state is regulated to minimize the cost functional

$$J_i\left(X(0),u_1^*,\ldots,u_{i-1}^*,u_i,u_{i+1}^*,\ldots,u_N^*,\omega^*\right) = \int_0^T\left(\|e_i\|^2 + \|u_i\|^2\right)\mathrm{d}t \quad (4)$$

where the first term $\|e_i\|^2$ represents a running cost and the second term represents a penalty on the control effort of the i-th agent, $i = 1,\ldots,N$;

(C1) The disturbance is attenuated by γ with respect to the mean-square error

$$\left(\|Z\|^2 + \sum_{i=1}^N \|u_i\|^2\right),$$

for $0 < \gamma < 1$. Namely,

$$\int_0^T\left(\|Z\|^2 + \sum_{i=1}^N \|u_i\|^2\right)\mathrm{d}t \le \gamma^2\int_0^T \|\omega\|^2\,\mathrm{d}t$$

for any $\omega \in \mathcal{L}_2$, $\omega \neq 0$.

Condition (C1) represents an individual optimality criterion for each agent, whereas Condition (C2) represents a robustness criterion for the global system.

Problem 1 can be interpreted as a multi-player version of the mixed H_2/H_∞ control problem and, following the approach of [18], it can be recast as a nonzero-sum differential game with $(N+1)$ players. To this end, let

$$J_{N+1}\left(X(0),u_1,\ldots,u_N,\omega\right) = \int_0^T\left(\gamma^2\|\omega\|^2 - \|Z\|^2 - \sum_{i=1}^N \|u_i\|^2\right)\mathrm{d}t. \quad (5)$$

Problem 2: nonzero-sum differential game formulation
Consider System (2). Determine a set of feedback strategies

$$\mathcal{U}^* = \left(u_1^*,\ldots,u_N^*,w^*\right)$$

that renders the zero equilibrium of System (2) stable in closed-loop with $\mathcal{U}^*$ and that satisfies the Nash equilibrium inequalities

$$J_i\left(X(0),\mathcal{U}^*\right) \le J_i\left(X(0),\mathcal{U}_{u_i}\right), \quad (6)$$

and

$$J_{N+1}\left(X(0),\mathcal{U}^*\right) \le J_{N+1}\left(X(0),\mathcal{U}_\omega\right), \quad (7)$$

where $\mathcal{U}_{u_i} = \left(u_1^*,\ldots,u_{i=1}^*,u_i,u_{i+1}^*\ldots,u_N^*,\omega^*\right)$ with $u_i \neq u_i^*$, for $i = 1,\ldots,N$, and $\mathcal{U}_\omega = \left(u_1^*,\ldots,u_N^*,\omega\right)$, with $\omega \neq \omega^*$, are sets of stabilizing feedback strategies.

The control strategies $\mathcal{U}^*$, namely the control inputs u_i^*, $i = 1,\ldots,N$, and disturbance w^* satisfying (6), $i = 1,\ldots,N$ and (7), constitute the Nash equilibrium strategies of the differential game in Problem 2. Considering the Nash equilibrium inequalities, it is clear that (6), $i = 1,\ldots,N$, correspond to Condition (C1), $i = 1,\ldots,N$, of Problem 2. Moreover, if $J_{N+1}\left(X(0),\mathcal{U}^*\right) > 0$, it follows from (7) that $J_{N+1}\left(X(0),\mathcal{U}_\omega\right) > 0$, for all $\omega \in \mathcal{L}_2$, thus satisfying Condition (C2) of Problem 2.

Exact solutions to the nonzero-sum differential game

Problem 2 constitutes a nonzero-sum differential game for which solutions, found using the dynamic programming method, are characterized by coupled Riccati differential equations (in the finite-horizon case) or coupled algebraic Riccati equations (in the infinite-horizon case). For more details on linear quadratic differential games see, for instance, [5], [35]. The game theoretic formulation in Problem 2 is particularly appealing because it naturally captures the trade-off between optimality and robustness (see, for instance, [1]). A solution of Problem 2 (considering linear feedback strategies only[1]) is provided in the following.

Assumption 1
The global state is constructed in a manner such that the running costs and terminal costs can be written as

$$q_i(X) \triangleq \|e_i\|^2 = X^{\mathrm{T}}Q_iX,$$

where $Q_i = Q_i^{\mathrm{T}} \ge 0$, for $i = 1,\ldots,N$.

Clearly, a consequence of Assumption 1 is that $\|Z\|^2$ can be written in the form

$$q_{N+1}(X) \triangleq \|Z\|^2 = X^{\mathrm{T}}Q_{N+1}X$$

where $Q_{N+1} = \sum_{i=1}^N Q_i$.

Proposition 1
Consider the global system (2), and the cost functionals (4), $i = 1,\ldots,N$, and (5). Suppose that we can find $P_i : \mathbb{R} \to \mathbb{R}^{Nn\times Nn}$, such that $P_i(t) = P_i(t)^{\mathrm{T}} \ge 0$, $i = 1,\ldots,N$, and $P_{N+1}(t) = P_{N+1}(t)^{\mathrm{T}} \le 0$ satisfying the coupled Riccati differential equations

$$\begin{aligned}
-\dot{P}_i(t) &= Q_i + P_i(t)A^g + A^{g\mathrm{T}}P_i(t) - P_i(t)B_i^gB_i^{g\mathrm{T}}P_i(t) \\
&- \sum_{j=1,j\neq i}^{N}\left(P_i(t)B_j^gB_j^{g\mathrm{T}}P_j(t) + P_j(t)Bj^gB_j^{g\mathrm{T}}P_i(t)\right) \\
&-\gamma^{-2}P_i(t)B_{N+1}^gB_{N+1}^{g\ \mathrm{T}}P_{N+1}(t) \\
&-\gamma^{-2}P_{N+1}(t)B_{N+1}^gB_{N+1}^{g\ \mathrm{T}}P_i(t) \\
P_i(T) &= 0
\end{aligned} \quad (8)$$

for $i = 1,\ldots,N$, and

[1] In general, linear quadratic nonzero-sum differential games can admit *nonlinear* solutions (see, for instance, [2]). However, as is commonly done (see, for instance, [13]), only linear feedback strategies are considered here.

$$
\begin{aligned}
&-\dot{P}_{N+1}(t) = -Q_{N+1} + P_{N+1}(t)A^g + A^{g\mathrm{T}}P_{N+1}(t) \\
&-\gamma^{-2}P_{N+1}(t)B^g_{N+1}B^g_{N+1}{}^{\mathrm{T}}P_{N+1}(t) \\
&-\sum_{j=1}^{N}\left(P_j(t)B^g_jB^{g\mathrm{T}}_jP_j(t) - P_{N+1}(t)B^g_jB^{g\mathrm{T}}_jP_j(t)\right) \\
&-\sum_{j=1}^{N}P_j(t)B^g_jB^{g\mathrm{T}}_jP_{N+1}(t) \\
&P_{N+1}(T) = 0
\end{aligned} \tag{9}
$$

Then, the following statements hold:

i. The Nash equilibrium strategies are given by

$$
\begin{aligned}
u_i^* &= -B_i^{g\mathrm{T}}P_i(t)X \\
\omega^* &= -\gamma^{-2}B^g_{N+1}{}^{\mathrm{T}}P_{N+1}(t)X
\end{aligned} \tag{10}
$$

for $i = 1,\ldots,N$;

ii. In the case that $u = u^*$ and $X(0) = 0$, Condition (C2) of Problem 2 is satisfied for any continuous function $\omega \in \mathcal{L}_2$.

Proof

Proposition 1 is essentially a multi-player version of the result in ([18], Theorem 2.1) and, as such, the proof is similar to the proof of the sufficient conditions provided therein. The proof consists of two main steps in which we demonstrate that Claims (i) and (ii) hold true, respectively.

As in [18], the statement (i) can be demonstrated by completion of squares[2]. Let us consider first the cost functionals (4), $i = 1,\ldots,N$.

From the boundary condition $P_i(T) = 0$ it follows that

$$
\begin{aligned}
&J_i\left(X(0),u_1,\ldots,u_N,\omega\right) - X(0)P_i(0)X(0) \\
&= \int_0^T X^{\mathrm{T}}Q_iX + \|u_i\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(X^{\mathrm{T}}P_iX\right)\mathrm{d}t.
\end{aligned} \tag{11}
$$

Substituting the system dynamics (2) and the time derivative of P_i given in (8), the above relation is transformed into

$$
\begin{aligned}
&J_i\left(X(0),u_1,\ldots,u_N,\omega\right) - X(0)P_i(0)X(0) \\
&= \int_0^T \left(\begin{aligned}&\|u_i - u_i^*\|^2 + 2\sum_{j=1,j\neq i}^{N}X^{\mathrm{T}}P_iB_j^g\left(u_j - u_j^*\right) \\ &+2X^{\mathrm{T}}P_iB^g_{N+1}\left(\omega - \omega^*\right)\end{aligned}\right)\mathrm{d}t
\end{aligned} \tag{12}
$$

To demonstrate that u_i^* is the Nash equilibrium strategy of the i-th agent, note that

$$
J_i\left(X(0),\mathcal{U}_{u_i}\right) - X(0)P_i(0)X(0) = \int_0^T \|u_i - u_i^*\|^2\,\mathrm{d}t,
$$

which is minimized when $u_i = u_i^*$. Namely, u_i^* is such that Inequality (6) is satisfied, for $i = 1,\ldots,N$. Considering the cost functional (5), following the same steps, it can be shown that

$$
\begin{aligned}
&J_{N+1}\left(X(0),u_1,\ldots,u_N,\omega\right) - X(0)^{\mathrm{T}}P_{N+1}(0)X(0) \\
&= \int_0^T \gamma^2\|\omega - \omega^*\|^2 - \sum_{j=1}^{N}\left(\|u_j^*\|^2 - \|u_j\|^2\right) \\
&\quad + 2\sum_{j=1}^{N}\left(X^{\mathrm{T}}P_{N+1}B_j\left(u_j - u_j^*\right)\right)\mathrm{d}t.
\end{aligned}
$$

Once again, it follows that

$$
\begin{aligned}
J_{N+1}\left(X(0),\mathcal{U}_\omega\right) - X(0)^{\mathrm{T}}P_{N+1}(0)X(0) \\
= \gamma^2\int_0^T \|\omega - \omega^*\|^2\,\mathrm{d}t,
\end{aligned}
$$

is minimized when $\omega = \omega^*$, *i.e.* ω^* satisfies Inequality (7), thus completing the proof of the statement (i).

The second part of the claim is demonstrated by noting that

$$
J_{N+1}\left(X(0),\mathcal{U}^*\right) = X(0)^{\mathrm{T}}P_{N+1}(0)X(0)
$$

Thus, for the initial condition $X(0) = 0$, the cost associated with the worst-case disturbance is zero, *i.e.*, $J_{N+1}(X(0),\mathcal{U}^*) = 0$. Therefore, it follows from (7) that any disturbance $\omega \in \mathcal{L}_2$ is such that

$$
J_{N+1}\left(X(0),\mathcal{U}_\omega\right) \geq 0.
$$

Condition (C2) then follows from the definition of the cost functional (5), which concludes the proof.

Remark 1

The so-called value functions

$$
V_i\left(X(t)\right) = X(t)^{\mathrm{T}}P_i(t)X(t),
$$

for $i = 1,\ldots,N+1$, are such that

$$
J_i\left(X(0),\mathcal{U}^*\right) = X(0)^{\mathrm{T}}P_i(0)X(0).
$$

Noting that

$$
J_i\left(X(0),\mathcal{U}^*\right) \geq 0,
$$

for $i = 1,\ldots,N$, it is clear that $P_i(t) \geq 0$ for $t \geq 0$. Similarly, since

$$
\begin{aligned}
&J_{N+1}\left(X(0),\mathcal{U}^*\right) = \int_0^T \gamma^2\|\omega^*\|^2\,\mathrm{d}t - \sum_{i=1}^{N}J_i\left(X(0),\mathcal{U}^*\right) \\
&\leq J_{N+1}\left(X(0),u_1^*,\ldots,u_N^*,0\right) \\
&= -\sum_{i=1}^{N}J_i\left(X(0),u_1^*,\ldots,u_N^*,0\right) \leq 0,
\end{aligned}
$$

it is clear that $P_{N+1}(t) \leq 0$ for $t \geq 0$.

In the infinite-horizon case, *i.e.*, in the limit as $T \to \infty$, the Nash equilibrium solution (10) requires the solution of coupled *algebraic* Riccati equations (AREs) instead of the coupled Riccati differential equations (8), $i = 1,\ldots,N$, and (9).

[2] Alternatively, the property in (i) can be demonstrated by applying the dynamic Programming principle (see, *e.g.* [7]).

Proposition 2

Consider the global system (2) and suppose that we can obtain a solution $P_i = P_i^{\mathrm{T}} \geq 0$, $P_{N_1} = P_{N_1}^{\mathrm{T}} \leq 0$ of the (static) coupled AREs

$$\begin{aligned} & Q_i + P_i A^g + A^{g\mathrm{T}} P_i - P_i B_i^g B_i^{g\mathrm{T}} P_i \\ & - \sum_{j=1, j\neq i}^{N} (P_i B_j^g B_j^{g\mathrm{T}} P_j + P_j Bj^g B_j^{g\mathrm{T}} P_i) \\ & -\gamma^{-2} P_i B_{N+1}^g B_{N+1}^g{}^{\mathrm{T}} P_{N+1} \\ & -\gamma^{-2} P_{N+1} B_{N+1}^g B_{N+1}^g{}^{\mathrm{T}} P_i = 0 , \end{aligned} \tag{13}$$

for $i = 1, \ldots, N$, and

$$\begin{aligned} & -Q_{N+1} + P_{N+1} A^g + A^{g\mathrm{T}} P_{N+1} \\ & -\gamma^{-2} P_{N+1} B_{N+1}^g B_{N+1}^g{}^{\mathrm{T}} P_{N+1} - \sum_{j=1}^{N} P_j B_j^g B_j^{g\mathrm{T}} P_j \\ & -\sum_{j=1}^{N} \left(P_{N+1} B_j^g B_j^{g\mathrm{T}} P_j + P_j B_j^g B_j^{g\mathrm{T}} P_{N+1} \right) = 0. \end{aligned} \tag{14}$$

Then, the following statements hold:

i. If the communication graph $\mathcal{G}$ is such that

$$\sum_{i=1}^{N} Q_i > 0 . \tag{15}$$

Then, the origin of System (2) in closed-loop with the feedback strategies

$$\begin{aligned} u_i^* &= -B_i^{g\mathrm{T}} P_i X , \\ \omega^* &= -\gamma^{-2} B_{N+1}^g{}^{\mathrm{T}} P_{N+1} X , \end{aligned} \tag{16}$$

for $i = 1, \ldots, N$, with P_i, $i = 1, \ldots, N+1$, satisfying (13), $i = 1, \ldots, N$, and (14), is stable;

ii. The Nash equilibrium strategies corresponding to Problem 2 in the infinite horizon case, *i.e.,* in the limit as $T \to \infty$, are given by (16), for $i = 1, \ldots, N$.

Proof

The proof essentially consists of two steps. To demonstrate stability of the closed-loop system, *i.e.,* statement (i) of the proposition, note that the summation of the N first AREs (13), $i = 1, \ldots, N$, yields the relation

$$\sum_{j=1}^{N} \left(P_j A_{cl}^g + A_{cl}^{g\mathrm{T}} P_j \right) + \sum_{j=1}^{N} \left(Q_j + P_j B_j^g B_j^{g\mathrm{T}} P_j \right) = 0 \tag{17}$$

where A_{cl}^g is the matrix describing the closed-loop system, namely $A_{cl}^g = A^g - \sum_{j=1}^{N} B_j^g B_j^{g\mathrm{T}} P_j - \gamma^{-2} B_{N+1}^g B_{N+1}^g{}^{\mathrm{T}} P_{N+1}$. Let $V_i(X) = X^{\mathrm{T}} P_i X$ denote the *value function* associated with the *i*-th agent (as in Remark 1), for $i - 1, \ldots, N$, and let $W(X) = \sum_{j=1}^{N} V_i$. Note that

$$W(X(0)) = \sum_{j=1}^{N} J_i \left(X(0), \mathcal{U}^* \right) > 0 ,$$

by Assumption (15). Moreover, along the trajectories of the closed-loop system the time derivative of the function W is given by

$$\dot{W} = X^{\mathrm{T}} \left(A_{cl}^{g\mathrm{T}} \sum_{j=1}^{N} P_j + \sum_{j=1}^{N} P_j A_{cl}^g \right) X .$$

It follows from (17) that

$$\dot{W} = -\sum_{j=1}^{N} X^{\mathrm{T}} \left(Q_j + P_j B_j^g B_j^{g\mathrm{T}} P_j \right) X \leq -\sum_{j=1}^{N} X^{\mathrm{T}} Q_j X .$$

Thus, from Assumption (15), $\dot{W} < 0$ for all $X \neq 0$ and stability of the closed-loop system follows from standard Lyapunov arguments.

The second statement follows directly from the same arguments in the proof of Proposition 1, noting that $\lim_{t\to\infty} X(t) = 0$ by stability. Consequently, the relations (11), $i = 1, \ldots, N$, and (12) hold with P_i, $i = 1, \ldots, N+1$, the (static) solutions of (13), $i = 1 \ldots, N$, and (14). Consider first the cost functionals (4), $i = 1, \ldots, N$. It can be shown, using (13), that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \left(X^{\mathrm{T}} P_i X \right) = -X^{\mathrm{T}} Q_i X - \| u_i \|^2 + \| u_i - u_i^* \|^2 \\ & + \sum_{j=1, j\neq i}^{N} \left(X^{\mathrm{T}} P_i B_j^g \left(u_j - u_j^* \right) + \left(u_j - u_j^* \right) B_j^{g\mathrm{T}} P_i X \right) \\ & + X^{\mathrm{T}} P_i B_{N+1}^g \left(\omega - \omega^* \right) + \left(\omega - \omega^* \right)^{\mathrm{T}} B_{N+1}^g{}^{\mathrm{T}} P_i X , \end{aligned}$$

for $i = 1 \ldots, N$. Thus, it follows (as in the finite-horizon case) that

$$J_i \left(X(0), \mathcal{U}_{u_i} \right) - X(0)^{\mathrm{T}} P_i X(0) = \int_0^T \| u_i - u_i^* \|^2 \, \mathrm{d}t$$

for $i = 1, \ldots, N$. Similar considerations for the cost functional (5), yield (as in the finite-horizon case)

$$\begin{aligned} & J_{N+1} \left(X(0), \mathcal{U}_{\omega} \right) - X(0)^{\mathrm{T}} P_{N+1} X(0) \\ & \qquad = \gamma^2 \int_0^T \| \omega - \omega^* \|^2 \, \mathrm{d}t , \end{aligned}$$

which demonstrates statement (ii) and thus completes the proof.

Remark 2

The condition (15) is standard in the context of infinite-horizon differential games (and infinite-horizon optimal control) as seen, for instance, in [5, 37]. The connection between this condition and the topology of the underlying graph $\mathcal{G}$ will be investigated in future work.

Approximate solutions to the nonzero-sum differential game

Solutions to coupled AREs (such as (13), $i = 1, \ldots, N$, and (14)) which arise in the context of linear quadratic differential games (see, for example [35], [13]) are oftentimes difficult to obtain. Coupled AREs have, for instance, been considered in [18], [15], [3] and solutions for particular classes of problems have been provided in [29], [30], [13], [14], [23]. In [24] it has been demonstrated that solving algebraic Riccati *inequalities* instead of *equalities* yields an approximate solution – in terms of a so-called ε_α-Nash equilibrium solution – to a differential game. The space of solutions of the inequalities contain the solutions of the original equalities and are, as a consequence, sometimes more readily solved. The notion of α-admissible strategies (introduced in [24]) is recalled and approximate solutions, *similar* to α-admissible strategies, are provided for Problem 2 in what follows. Solving inequalities instead of (13), $i=1,\ldots,N$ and (14) can be interpreted as solving a differential game subject to "modified cost functionals". While in general the results may differ significantly from the original game, the notion of an ε_α-Nash equilibrium solution enables us to relate the resulting feedback strategies to the original game.

Definition 1 [24]
A set of linear[3] state feedback control inputs

$$\mathcal{U}=(u_1,\ldots,u_N,\omega)$$

is said to be α-admissible, with $\alpha>0$, if the origin of System (2) in closed-loop with $\mathcal{U}$ is such that

$$\sigma\left(\bar{A}_{cl}+\alpha I\right)\in\mathbb{C}^-,$$

where $\bar{A}_{cl}$ is the matrix describing the closed-loop system.

Proposition 3
Consider the global system (2) and suppose that we can obtain $P_i=P_i^{\mathrm{T}}\geq 0$, $i-1,\ldots,N$, and $P_{N+1}=P_{N+1}^{\mathrm{T}}\leq 0$ satisfying the inequalities

$$\begin{aligned}&Q_i+P_iA^g+A^{g\mathrm{T}}P_i-P_iB_i^gB_i^{g\mathrm{T}}P_i\\&-\sum_{j=1,j\neq i}^{N}\left(P_iB_j^gB_j^{g\mathrm{T}}P_j+P_jBj^gB_j^{g\mathrm{T}}P_i\right)\\&-\gamma^{-2}P_iB_{N+1}^gB_{N+1}^{g\;\mathrm{T}}P_{N+1}\\&-\gamma^{-2}P_{N+1}B_{N+1}^gB_{N+1}^{g\;\mathrm{T}}P_i\leq 0,\end{aligned}\tag{18}$$

for $i=1,\ldots,N$, and

$$\begin{aligned}&-Q_{N+1}+P_{N+1}A^g+A^{g\mathrm{T}}P_{N+1}\\&-\gamma^{-2}P_{N+1}B_{N+1}^gB_{N+1}^{g\;\mathrm{T}}P_{N+1}-\sum_{j=1}^{N}P_jB_j^gB_j^{g\mathrm{T}}P_j\\&-\sum_{j=1}^{N}\left(P_{N+1}B_j^gB_j^{g\mathrm{T}}P_j+P_jB_j^gB_j^{g\mathrm{T}}P_{N+1}\right)\geq 0.\end{aligned}\tag{19}$$

Moreover, suppose that the communication graph $\mathcal{G}$ is such that (15) holds. Then, the set of feedback strategies $\mathcal{U}^*=\left(u_1^*,\ldots,u_N^*,\omega^*\right)$, with u_i^* and ω^* given by (16), $i=1,\ldots,N$, with P_i and P_{N+1} satisfying (18), $i=1,\ldots,N$, and (19), respectively, are such that the following statements hold:

i. System (2) in closed-loop with $\mathcal{U}^*$ is stable.
ii. Considering the infinite-horizon case, *i.e.,* in the limit as $T\to\infty$, the inequalities

$$J_i\left(X(0),\mathcal{U}^*\right)\leq J_i\left(X(0),\mathcal{U}_{u_i}\right)+\varepsilon_\alpha\tag{20}$$

are satisfied with $\varepsilon_\alpha>0$, parameterized in $\alpha>0$ and $X(0)$, for any α-admissible set of strategies $\mathcal{U}_{u_i}=\left(u_1^*,\ldots,u_{i-1}^*,u_i,u_{i+1}^*,\ldots,u_N^*,\omega^*\right)$, for any $\alpha>0$ and for $i=1,\ldots,N$.

iii. In the case where $u_i=u_i^*$ and $X(0)=0$, Condition (C2) of Problem 2 is satisfied for any continuous function $\omega\in\mathcal{L}_2$.

Proof
Stability can be demonstrated following the same steps used in the first part of the proof of Proposition 2. The statement (ii) can be demonstrated following steps similar to those provided in [24], Proposition 2. Namely, consider the inequalities (18), $i=1,\ldots,N$. These inequalities imply that there exist matrices $\Upsilon_i=\Upsilon_i^{\mathrm{T}}\geq 0$, such that

$$\begin{aligned}&Q_i+\Upsilon_i+P_iA^g+A^{g\mathrm{T}}P_i-P_iB_i^gB_i^{g\mathrm{T}}P_i\\&-\sum_{j=1,j\neq i}^{N}\left(P_iB_j^gB_j^{g\mathrm{T}}P_j+P_jBj^gB_j^{g\mathrm{T}}P_i\right)\\&-\gamma^{-2}P_iB_{N+1}^gB_{N+1}^{g\;\mathrm{T}}P_{N+1}\\&-\gamma^{-2}P_{N+1}B_{N+1}^gB_{N+1}^{g\;\mathrm{T}}P_i=0,\end{aligned}$$

for $i=1,\ldots,N$. Similarly, Inequality (19) implies that there exists a matrix $\Upsilon_{N+1}=\Upsilon_{N+1}^{\mathrm{T}}\geq 0$ such that

$$\begin{aligned}&-Q_{N+1}-\Upsilon_{N+1}+P_{N+1}A^g+A^{g\mathrm{T}}P_{N+1}\\&-\gamma^{-2}P_{N+1}B_{N+1}^gB_{N+1}^{g\;\mathrm{T}}P_{N+1}-\sum_{j=1}^{N}P_jB_j^gB_j^{g\mathrm{T}}P_j\\&-\sum_{j=1}^{N}\left(P_{N+1}B_j^gB_j^{g\mathrm{T}}P_j+P_jB_j^gB_j^{g\mathrm{T}}P_{N+1}\right)=0.\end{aligned}$$

It follows that the feedback strategies $\mathcal{U}^*$ are the Nash equilibrium strategies of a nonzero-sum differential game with the *modified* cost functionals

$$\tilde{J}_i\left(X(0),\mathcal{U}\right)=J_i\left(X(0),\mathcal{U}\right)+\int_0^\infty X^{\mathrm{T}}\Upsilon_iX\mathrm{d}t,\tag{21}$$

for $i=1,\ldots,N$, and

$$\tilde{J}_{N+1}\left(X(0),\mathcal{U}\right)=J_{N+1}\left(X(0),\mathcal{U}\right)-\int_0^\infty X^{\mathrm{T}}\Upsilon_{N+1}X\mathrm{d}t,\tag{22}$$

wherein $\mathcal{U}=\left(u_1,\ldots,u_N,\omega\right)$. Let $\hat{X}(t)$ denote the trajectory of System (2) in closed-loop with the α-admissible set of strategies $\hat{\mathcal{U}}_{u_i}=\left(u_1^*,\ldots,u_{i-1}^*,\hat{u}_i,u_{i+1}^*,\ldots,u_N^*,\omega^*\right)$, where $\hat{u}_i=K_iX$ is such that the closed-loop system has the minimum possible decay rate (specified by $\alpha>0$). It is straightforward to see that

$$J_i\left(X(0),\mathcal{U}^*\right)\leq\tilde{J}_i\left(X(0),\mathcal{U}^*\right)\leq\tilde{J}_i\left(X(0),\hat{\mathcal{U}}_{u_i}\right),$$

since $\Upsilon_i\geq 0$, $i=1,\ldots,N$. Namely, the relation

$$J_i\left(X(0),\mathcal{U}^*\right)\leq J_i\left(X(0),\hat{\mathcal{U}}_{u_i}\right)+\int_0^\infty\hat{X}^{\mathrm{T}}\Upsilon_i\hat{X}\mathrm{d}t\tag{23}$$

holds and, exploiting α-admissibility of the set of strategies $\hat{\mathcal{U}}_{u_i}$ the second term on the right-hand side of (23), which accounts for an *additional running cost*, can be bounded from above. To this end, let

3 While we limit our at tention to linear feedback strategies, the notion can be defined for general (possibly nonlinear) strategies as in [24].

$\hat{A}_{cl,\alpha}$ denote the matrix describing System (2) in closed-loop with $\hat{\mathcal{U}}_{u_i}$ and note that since $\hat{\mathcal{U}}_{u_i}$ is α-admissible, the Lyapunov equation

$$\hat{P}_{i,\alpha}\hat{A}_{cl,\alpha} + \hat{A}_{cl,\alpha}^{\mathrm{T}}\hat{P}_{i,\varepsilon} + \Upsilon_i = 0,$$

has a unique solution $\hat{P}_{i,\varepsilon} = \hat{P}_{i,\varepsilon}^{\mathrm{T}} \geq 0$. Moreover, the function $\hat{V}_i = \hat{X}^{\mathrm{T}}\hat{P}_{i,\varepsilon}\hat{X}$ is such that

$$\dot{\hat{V}}_i(\hat{X}) = \hat{P}_{i,\varepsilon}\hat{A}_{cl,\alpha} + \hat{A}_{cl,\alpha}^{\mathrm{T}}\hat{P}_{i,\varepsilon} = -\hat{X}\Upsilon_i\hat{X}.$$

Integrating both sides of this relation from zero to infinity (noting that $\lim_{t\to\infty} X(t) = 0$ since $\hat{\mathcal{U}}_{u_i}$ is α-admissible), yields

$$\hat{V}(\hat{X}(0)) = \hat{X}(0)^{\mathrm{T}}\hat{P}_{i,\varepsilon}\hat{X}(0) = \int_0^\infty \hat{X}\Upsilon_i\hat{X}\,\mathrm{d}t.$$

It follows from (23) that

$$J_i(X(0),\mathcal{U}^*) \leq J_i(X(0),\hat{\mathcal{U}}_{u_i}) + X(0)^{\mathrm{T}}\max_i\{P_{i,\varepsilon}\}X(0),$$

$i = 1,\ldots,N$. The modified cost functional (22), on the other hand, is such that

$$\tilde{J}_{N+1}(X(0),\mathcal{U}^*) = X(0)^{\mathrm{T}}P_{N+1}X(0) \leq 0.$$

It follows that

$$\tilde{J}_{N+1}(0,\mathcal{U}_\omega) \geq \tilde{J}_{N+1}(0,\mathcal{U}^*) = 0,$$

which implies that the inequality

$$\begin{aligned}
&\gamma^2\int_0^\infty \|\omega\|^2\,\mathrm{d}t \\
&\geq \int_0^\infty\left(\|Z\|^2 + \sum_{i=1}^N\|u_i^*\|^2\right)\mathrm{d}t + \int_0^\infty X^{\mathrm{T}}\Upsilon_{N+1}X\mathrm{d}t \\
&\geq \int_0^\infty\left(\|Z\|^2 + \sum_{i=1}^N\|u_i\|^2\right),
\end{aligned}$$

is satisfied, for any $\omega \in \mathcal{L}_2$ when $X(0) = 0$, which concludes the proof.

Remark 3

Interestingly, considering Problem 2, the results in Proposition 3 entail that the "optimality criteria" in (C1) are solved approximately, whereas the "robustness" criterion in (C2) is solved exactly. This result is in line with the observation that there is, in general, a trade-off between optimality and robustness.

Remark 4

A set of matrices P_i, $i = 1,\ldots,N$, satisfying the coupled AREs (13), $i = 1,\ldots,N$, and (14) also satisfy the inequalities (18), $i = 1,\ldots,N$, (19). That is, the space of solutions of the inequalities is larger – and includes – the space of solutions of the coupled AREs.

Simulations

In this section, we present two numerical examples to illustrate the theoretical results presented in the previous sections. The first example concerns a simple, scalar consensus problem, whereas the second example concerns formation control for a fleet of autonomous vehicles. While the two examples are considered separately in the following, both involve the same number of agents and communication graph.

Consensus at the origin

Consider first the case in which we wish to steer a group of agents towards a common consensus value which is fixed *a priori*. Without loss of generality, we consider this common value to be $x_i = 0$, $i = 1,\ldots,N$. Towards this end we construct the global state simply as the collection of the individual states of each agent, namely

$$X = \left[x_1^{\mathrm{T}}, x_2^{\mathrm{T}},\ldots,x_N^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{24}$$

resulting in the global system described by (2) with

$$A^g = \mathrm{blockdiag}\{A_1,\ldots,A_N\},$$

$$B_1^g = \left[B_1^{\mathrm{T}},0,\ldots,0\right]^{\mathrm{T}},\ldots,B_N^g = \left[0,\ldots,0,B_{N+1}^{\mathrm{T}}\right]^{\mathrm{T}},$$

and

$$B_{N+1}^g = \left[B_{N+1}^{\mathrm{T}},B_{N+1}^{\mathrm{T}},\ldots,B_{N+1}^{\mathrm{T}}\right]^{\mathrm{T}}.$$

Consider the case, similar to the one presented in [10], of four agents. Each agent $i \in \{1,\ldots,4\}$ is described by a scalar state $x_i \in \mathbb{R}$ and satisfies the dynamics

$$\dot{x}_i = B_i u_i + B_5\omega,$$

i.e., $A_i = 0$, for $i = 1,\ldots,N$, and consider the case in which the matrices B_i are identical for all $i = 1,\ldots,4$. The communication graph is described by the edge set

$$\mathcal{E} = \{(1,1);(1,2);(1,3);(2,2),(2,3);(3,3);(3,1);(3,4)\},$$

and the corresponding running costs for each agent i, $i = 1,\ldots,4$, are defined by the matrices

$$Q_1 = \begin{bmatrix} \frac{5}{4} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad Q_2 = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{5}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} \frac{1}{9} & 0 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{9} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{22}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\quad Q_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{5}{4} \end{bmatrix}.$$

Consider the case in which $\gamma = 0.5$, $B_i = 1$, for $i = 1,\ldots,N$, and $B_5 = 0.1$ and consider Problem 2 in the infinite-horizon case. The set of matrices

$$P_1 = \begin{bmatrix} 1.1287 & 0.0074 & -0.1640 & 0.0189 \\ 0.0074 & 0.0005 & 0.0020 & 0.0004 \\ -0.1640 & 0.0020 & 0.0672 & -0.0004 \\ 0.0189 & 0.0004 & -0.0004 & 0.0005 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.0888 & -0.2087 & 0.0038 & -0.0009 \\ -0.2087 & 1.1485 & 0.0047 & 0.0185 \\ 0.0038 & 0.0047 & 0.0007 & 0.0004 \\ -0.0009 & 0.0185 & 0.0004 & 0.0005 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.0407 & -0.0036 & -0.1142 & -0.0003 \\ -0.0036 & 0.0416 & -0.1054 & -0.0001 \\ -0.1142 & -0.1054 & 1.5944 & 0.0220 \\ -0.0003 & -0.0001 & 0.0220 & 0.0006 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 0.0006 & 0.0005 & 0.0025 & 0.0054 \\ 0.0005 & 0.0005 & 0.0020 & 0.0072 \\ 0.0025 & 0.0020 & 0.0672 & -0.1669 \\ 0.0054 & 0.0072 & -0.1669 & 1.1559 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} -1.2426 & 0.2205 & 0.2906 & -0.0062 \\ 0.2205 & -1.1750 & 0.1155 & -0.0088 \\ 0.2906 & 0.1155 & -1.7071 & 0.1654 \\ -0.0062 & -0.0088 & 0.1654 & -1.1390 \end{bmatrix},$$

constitutes a solution[4] of the coupled AREs (13), $i = 1,\ldots,N$, and (14). Note that the condition (15) is satisfied. The performance of the resulting feedback control laws u_i^* given in (16) is evaluated through a series of simulations. In all plots blue indicates Agent 1, green indicates Agent 2, purple indicates Agent 3 and cyan indicates Agent 4, whereas red indicates the quantities relating to the disturbance, *i.e., Player 5* in the differential game defined in Problem 2. Note that the running cost associated with Player 5 is given by $Q_5 = \sum_{i=1}^{N} Q_i$.

Consider first the case in which the state of the system is perturbed, such that the initial states are $x_1(0) = 1$, $x_2(0) = 2$, $x_3(0) = 0$ and $x_4(0) = -1$. Suppose that the system is influenced by a disturbance of the form $\omega = \omega_k = k\omega^*$, where $k \in \mathbb{R}$ is a constant parameter. Let $\mathcal{U}_{\omega_k}$ denote the set of strategies

$$\mathcal{U}_{\omega_k} = \left(u_1^*,\ldots,u_N^*,\omega_k\right).$$

[4] As mentioned in § "Approximate solutions", obtaining solutions of coupled Riccati equations arising in nonzero-sum differential games is not straight-forward, in general. In this numerical example, the solution of the coupled AREs has been obtained by numerically solving the finite-horizon equations (8), $i = 1,\ldots,4$, and (9) backwards in time using the function 'ode45" in MATLAB. In this particular example, the resulting values of $P_i(0)$, for $i = 1,\ldots,N+1$, converge to a solution of the AREs characterizing the solution of the differential game considered.

The cost $J_{N+1}\left(X(0),\mathcal{U}_{\omega_k}\right)$ obtained for various values of k, when all four agents apply the feedback control strategies u_i^* given in (16), $i = 1,\ldots,4$, is presented in Figure 1, where the minimum cost is indicated by the black diamond marker and corresponds to the value $k = 1$. Similarly, consider the four cases in which the system is influenced by the worst-case disturbance ω^* and each agent i, for $i = 1,\ldots,4$, adopts a control input of the form $u_i = u_{i,k} = ku_i^*$ (where $k \in \mathbb{R}$ is again a constant parameter) while all other agents j, $j = 1,\ldots,4$, $j \neq i$ adhere to the Nash equilibrium control laws u_j^*. Let $\mathcal{U}_{u_{i,k}}$ denote the set of strategies

$$\mathcal{U}_{u_{i,k}} = (u_1^*,\ldots,u_{i-1}^*,u_{i,k},u_{i+1}^*,\ldots,u_N^*,\omega^*).$$

The variations of the resulting costs, namely $J_i(X(0),\mathcal{U}_{i,k})$, with the parameter k are shown in Figure 2, for Agent 1 (top, left), Agent 2 (top, right), Agent 3 (bottom, left) and Agent 4 (bottom, right). In each plot the minimum cost is indicated by the black diamond marker and corresponds to $k = 1$. The latter result serves as an illustration that Condition (C1) of Problem 2 is satisfied by the Nash equilibrium solution of the differential game defined in Problem 2.

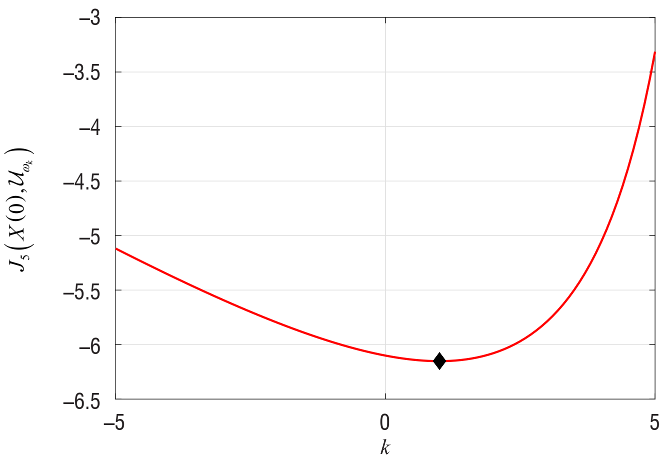

Figure 1 - Variation of $J_{N+1}\left(X(0),\mathcal{U}_{\omega_k}\right)$ as a function of k

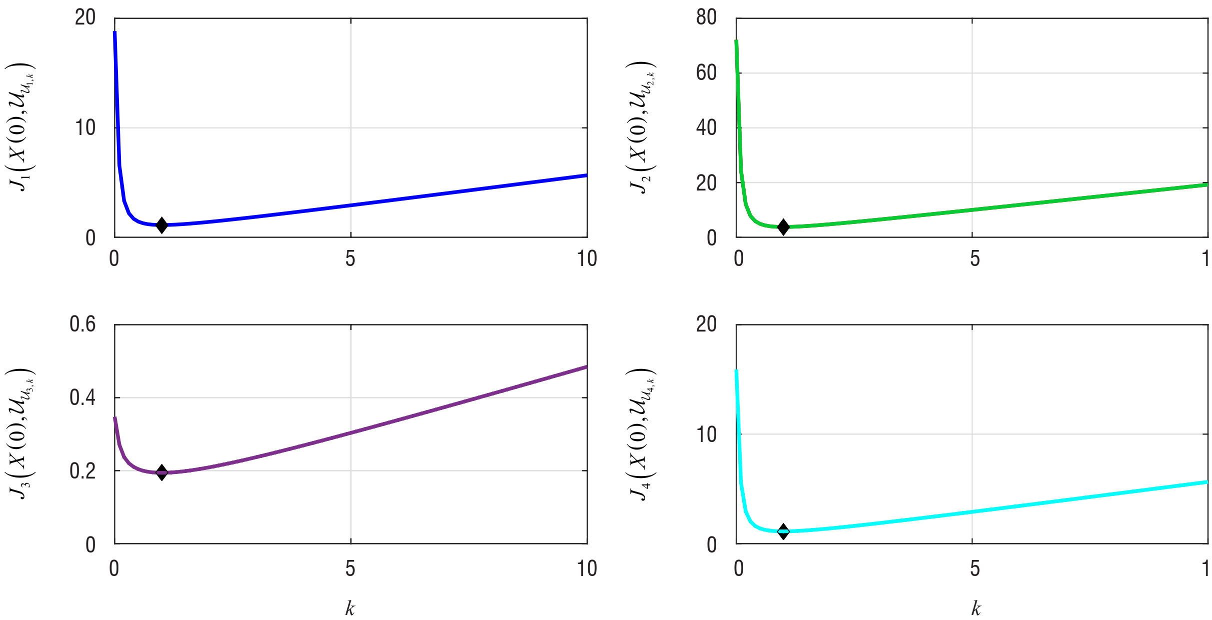

Figure 2 - The variations of $J_i\left(X(0),\mathcal{U}_{u_{i,k}}\right)$ as a function of k for Agent 1 (top, left), Agent 2 (top, right), Agent 3 (bottom, left) and Agent 4 (bottom, right).

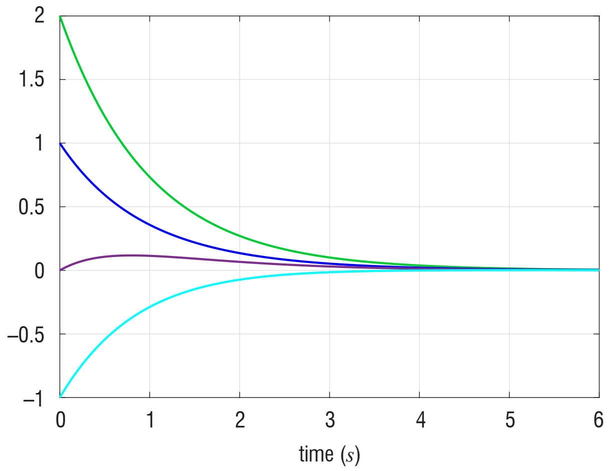

Figure 3 - The time histories of x_1 (blue line), x_2 (green line), x_3 (purple line) and x_4 (cyan line), when the disturbance and control inputs correspond to the Nash equilibrium solution of Problem 2.

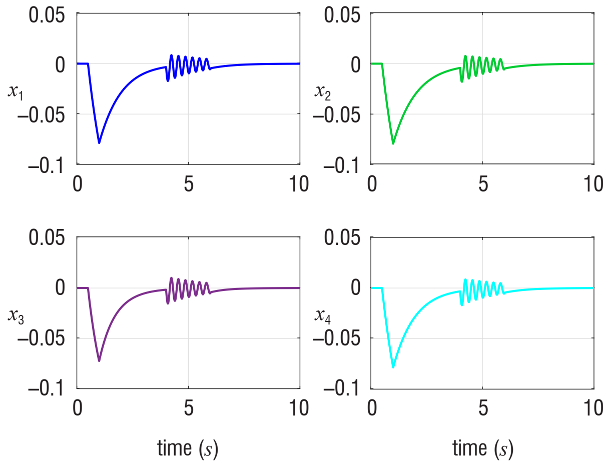

Figure 5 - Time histories of the states x_i, $i=1,\dots,4$.

The time histories of each individual state corresponding to the case in which $k=1$, *i.e.,* when $\omega=\omega^*$ and $u_i=u_i^*$, $i=1,\dots,4$, is shown in Figure 3.

Consider now the case in which the system starts in equilibrium[5], *i.e.,* $x_i(0)=0$, for $i=1,\dots,4$, and is subject to the disturbance

$$\omega=\begin{cases}-2 & \text{for } 0.5<t<1,\\ 20e^{-\frac{t}{2}}\sin(20t) & \text{for } 4<t<6,\\ 0 & \text{otherwise}\end{cases}$$

depicted in Figure 4. The resulting time histories of x_1 (top, left), x_2 (top, right), x_3 (bottom, left) and x_4 (bottom, right) are shown in Figure 5, whereas the time histories of the feedback control inputs u_1^* (top, left), u_2^* (top, right), u_3^* (bottom, left) and u_4^* (bottom, right) are shown in Figure 6. The time histories of the cost functionals (4) (top), for $i=1,\dots,4$, and (5) (bottom) are shown in Figure 7. Notably $J_5>0$ at all times, which indicates that the robustness property (C2) of Problem 2 is satisfied.

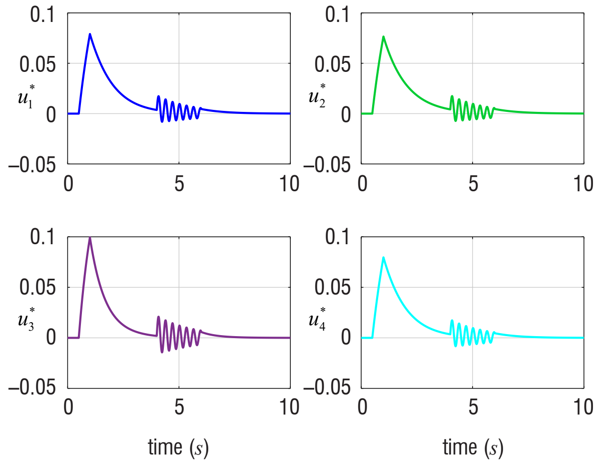

Figure 6 - Time histories of the control inputs u_i^*, $i=1,\dots,4$, of each agent in response to the disturbance ω.

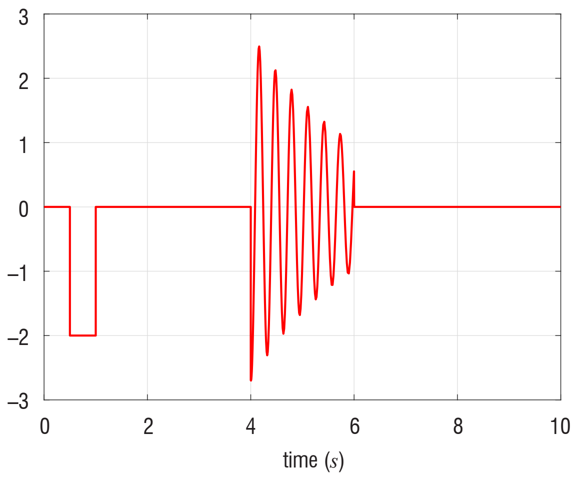

Figure 4 - Time history of the disturbance ω influencing the system.

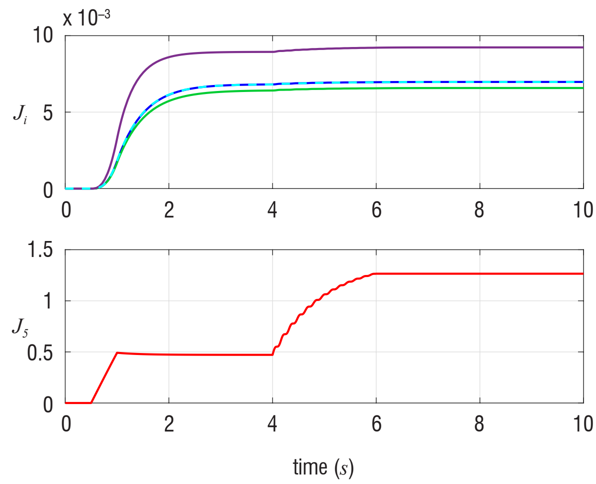

Figure 7 - Time histories of costs (4) (top), for $i=1,\dots,4$, and (5) (bottom).

5 Note that the worst-case disturbance $\omega^*(t)\equiv 0$ in this case.

Application to UAV formation flight

In this example, we consider four UAVs connected according to the same graph as in the previous example. The dynamics of the agents are modified to reflect the UAVs' behaviors and are described by the Euler-Lagrange system

$$M\ddot{q}_i + C\dot{q}_i = u_i + d \quad (25)$$

where $u_i \in \mathbb{R}^n$ is the control input of Agent i, $M \in \mathbb{R}^{n\times n}$ is the inertia matrix of Agent i, $C \in \mathbb{R}^{n\times n}$ is the matrix of the Coriolis and centripetal terms, and d is the additive external state perturbation (which is common to all agents). The values of M and C are considered identical for all agents. The state vector of each agent is defined as $(q_i, \dot{q}_i)$, which corresponds to the position and speed of the UAVs in some reference frame. In the following, we consider only the y and z variations of the positions, and assume that the component of the trajectory along the x axis is controlled separately and decoupled from the y and z evolution. The control objective is to drive the fleet to a desired target formation in some global reference frame $\mathcal{R}$. The target formation is represented via the relative coordinate vectors $r_{ij} = q_i - q_j$ between two agents i and j, and the target relative coordinate vector r_{ij}^* for all $(i,j) \in \mathcal{N}$. A target formation is defined by the set $\{r_{ij}^*, (i,j) \in \mathcal{N}\}$. Consider, without loss of generality, the first agent as a reference agent and introduce the target relative configuration vector $r^* = \left[r_{11}^{*T} \quad \dots \quad r_{1N}^{*T} \right]^T$. Any target relative configuration vector r_{ij}^* can be expressed as $r_{ij}^* = r_{1i}^* - r_{1j}^*$. The global formation problem can thus be expressed using the dynamic model

$$M\ddot{r}_{ij}^* + C\dot{r}_{ij}^* = u_i + d ,$$

for $i = 1, \dots, 4$. The control laws that ensure convergence to the consensus at the origin can be sought by using a consensus error expressed as in (3). Since we are considering the same communication graph as in the previous example, the running cost for each agent is defined by the matrices Q_i, $i = 1, \dots, 5$ presented in the previous subsection. Consider the case in which the matrices M and C are given by

$$M = \begin{bmatrix} 0.56 & -2.23 \\ -2.23 & 0.56 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.40 & -1.76 \\ -1.76 & 2.99 \end{bmatrix}.$$

The target formation has the shape of a square whose center evolves along the x axis. The feedback control laws u_i^* that drive the fleet to the sought formation are obtained, as in the previous example, via the solution of the coupled AREs. The resulting trajectories and the time histories of the speed vectors for the four agents in the presence of the worst-case disturbance are depicted in Figures 8 and 9, respectively. To assess the robustness properties of the approach, the set of trajectories and speed evolutions using the feedback control laws with random initial state values and disturbances equal to k times the worst-case disturbance, with $0 < k \le 5$ randomly chosen, are presented in Figures 10 and 11, respectively. The target formations are reached in all cases while the speed values converge to the consensus speed.

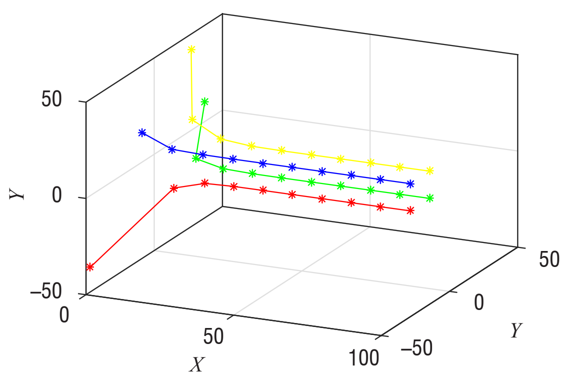

Figure 8 - Evolution of the four agents' trajectories obtained with the Nash equilibrium strategies.

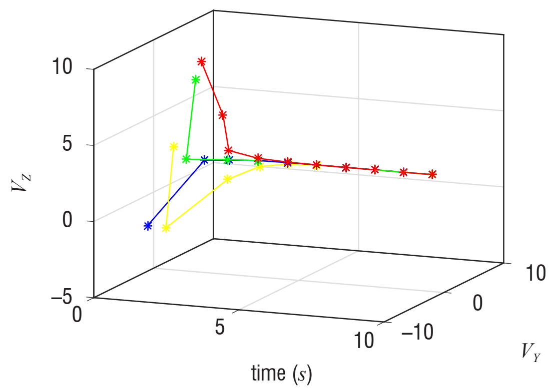

Figure 9 - Evolution of the four agents' speed values obtained with the Nash equilibrium solution.

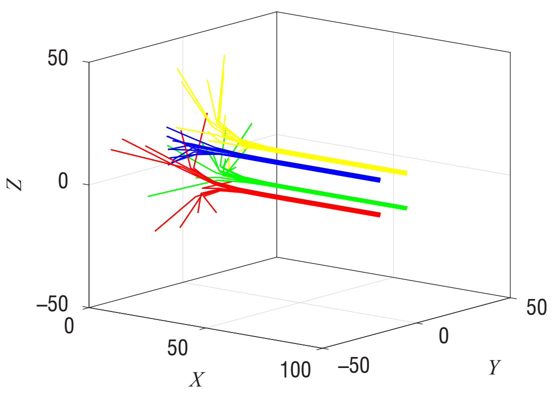

Figure 10 - Variations of the four agents' trajectories with random uncertainty.

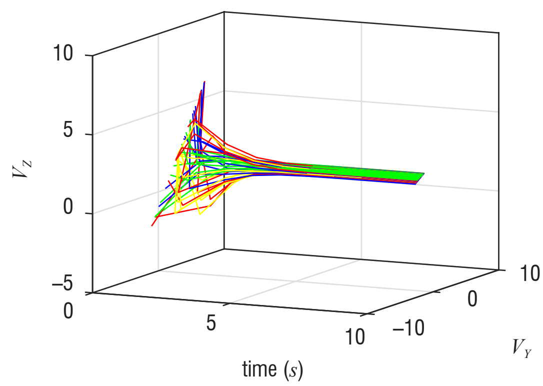

Figure 11 - Variations of the four agents' speed vectors with random uncertainty.

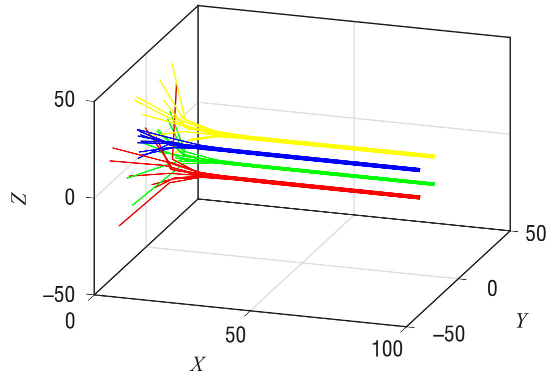

Figure 12 - Variations of the four agents' trajectories with perturbed dynamics.

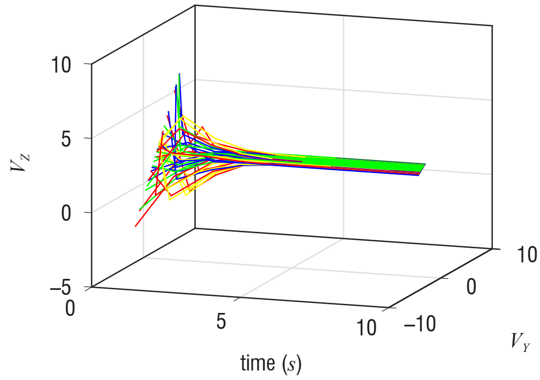

Figure 13 - Variations of the four agents' speed vectors with perturbed dynamics.

In order to evaluate the robustness of the approach to perturbations on the system dynamics, Matrices M and C have been replaced by perturbed matrices, where each element is perturbed with an additional uncertainty of at most 10% of the initial value. As illustrated in Figures 12 and 13, the formation still converges to the consensus.

Conclusion

The *robust consensus-seeking problem* is considered in this paper. Multi-agent systems in which each agent satisfies linear dynamics are considered, and the consensus problem is formulated as a multi-player nonzero-sum differential game. Exact solutions are provided for both finite-horizon and infinite-horizon problems, in terms of coupled Riccati equations. Motivated by the fact that coupled algebraic Riccati equations are, in general, difficult to solve, approximate solutions are provided for the latter. The results are demonstrated by means of two simulation studies. Directions for future research include the consideration of nonlinear systems. Moreover, it is of particular interest to consider distributed settings, in which the control inputs for each agent must be computed subject to communication and information constraints ■

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